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Boundary Conformal Field Theory and ribbon graphs: a tool for open/closed string dualities

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Abstract: We construct and fully characterize a scalar boundary conformal field theory on a triangulated Riemann surface. The results are analyzed from a string theory perspective as tools to deal with open/closed string dualities.

Keywords: Conformal Field Models in String Theory, Bosonic Strings, String Duality, Lattice Models of Gravity.

Contents

1. Introduction and motivations

The dynamical description of the open-to-closed worldsheet transition is a delicate issue in the analysis of open/closed string dualities. Simplicial techniques play, in this setting, an important role providing a deep, unexpected connection between Riemann moduli space, conformal field theory, and the study of the gauge/gravity correspondence [1 − 7]. The kinematical rationale motivating such a role is provided by the ribbon graph realization of gauge theory diagrams [4], by the Schwinger parametrization of the polytopal cells of the graph, and by Strebel's theorem [8, 9], connecting the combinatorics of decorated ribbon graphs to the conformal geometry of the worldsheet. Even if this suggests that we are disclosing some deep discrete structure underlying string dualities, we must stress that the dynamical aspects of the connection between combinatorial structures and dualities is not so obvious. Here, Boundary Conformal Field Theory (BCFT) is called into play in a rather sophisticated way: the ribbon graph, and more generally the underlying (metrically) triangulated surface, becomes the combinatorial pattern along which the quantum matter fields, described by cell-wise independent BCFTs, interact. One expects that this interaction generates a BCFT living on an open Riemann surface with gauge decorated boundaries living on D-branes which act as sources of the gauge fields. The actual realization of such a BCFT is notoriously difficult to carry out explicitly, and a careful analysis of its construction is the main motivation underlying the analysis presented in this paper. Whereas our set up will be necessarily rather elementary on the CFT side, since we consider bosonic matter fields, it will be geometrically quite general in the sense that we consider

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metric ribbon graphs which are dual to triangulated surfaces with curvature defects. The reason of this generality is that curvature defects provide a natural order parameter which allows to map closed N-pointed Riemann surfaces into open Riemann surfaces with N boundary components (of definite lengths). Such a mapping has been described in details in [10 − 12] where metric triangulations with variable connectivity and variable edge lengths, the Random Regge Triangulations (RRT), have been considered. It also has an equivalent description in terms of hyperbolic geometry [13], which is naturally activated if one consider matter in the form of twistorial fields. For simplicity, we limit here our analysis to the more elementary RRT case. In such a setting, a basic step is to exploit the fact that a RRT with curvature defects can be naturally uniformized on an open Riemann surface, with finite cylindrical ends whose moduli are provided by the defect [10]. These finite cylindrical ends are glued together along the pattern defined by the ribbon graph baricentrically dual to the parent triangulation. One can naturally interpret each cylindrical end as an open string connected at one boundary to the ribbon graph associated to the discretized worldsheet, while the other boundary lies on a D-brane acting as a source for gauge fields. The main topic we address here concerns the coherent description and definition of a BCFT - (i.e. an open string theory in a worldsheet meaning) on such a background. In particular, at a fixed genus g and at a fixed number of vertexes N_0 in the underlying simplicial complex, we first quantize a D-dimensional BCFT on single cylindrical end. Then, we will show how the resulting theories on different cylinders can be glued together along the intersection pattern defined by the ribbon graph associated to the given RRT (for some preliminary results in this direction see [14]). This latter aspect, which is the main result of this paper, calls into play BCFT in a non trivial way and it is based upon a careful use of the automorphisms of the chiral algebra naturally associated with the conformal theory. The generality of the overall construction allows us to take in account also open string gauge degrees of freedom, since the outer boundary of the cylindrical ends can lay on a stack of D-branes. The resulting decoration of each open string with an assignation of Chan-Paton factors provides a natural way to *dynamically* color the ribbon graph Γ with labels proper of the chosen gauge group, hence constructing out of Γ a genuine 't Hooft diagram. Moreover, if we consider toroidal compactifications for the target space of the bosonic scalar fields, the D-branes provide an explicit expression for the formal rules describing the Γ-interacting BCFTs on different cylinders. In particular, when the conformal field theory becomes rational, we can completely characterize the dynamic of a relevant class of fields. These fields play a key role in the description of the interactions between the different conformal theories on the cylinders. We refer to them as Boundary Insertion Operators and we provide a concrete description for both their analytic and algebraic structure.

Outline. This paper is conceptually divided into two parts. The first one, which covers section 2, is devoted to the construction of a formal amplitude on the discrete open surface M_∂. This is achieved coupling such a geometry with a scalar conformal field theory, and it involves the definition of Boundary Insertion Operators as mediators along the interaction pattern defined by the ribbon graph Γ associated with M_∂.

In the second part we provide an explicit prescription to dynamically couple the above geometry with an open string gauge theory in target space (see section 3). In this framework, in section 4, we provide an explicit characterization of the BCFT interaction scheme by discussing the associated amplitude.

2. Boundary Conformal Field Theory on M_∂

As a starting point let us summarize the properties of the specific geometric setup we shall deal with throughout the whole paper. Let M denote a closed 2-dimensional oriented manifold of genus g. A random Regge triangulation of M is an homeomorphism $|T_l| \to M$ where T_l denote a 2-dimensional semi-simplicial complex with underlying polyhedron $|T_l|$ and where each edge $\sigma^1(h,j)$ of T_l is realized by a rectilinear simplex of variable length $l(h,j)$. Note that the connectivity of T_l is not a priori fixed as in the case of standard Regge triangulations. Let $N_i(T_l) \in \mathbb{N}$ denote the number of i-dimensional subsimplices $\sigma^i(\ldots)$ of T_l. Consider the (first) barycentric subdivision $T_l^{(1)}$ of $|T_l| \to M$. The closed stars, in such a subdivision, of the vertices of the original triangulation $|T_l| \to M$ form a collection of 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T_l)}$ characterizing the *conical* Regge polytope $|P_{T_l}| \to M$ barycentrically dual to $|T_l| \to M$. Note that here we are considering a geometrical presentation $|P_{T_l}| \to M$ of P where the 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T_l)}$ retain the conical geometry induced on the barycentric subdivision by the original metric structure of $|T_l| \to M$. This latter is locally Euclidean everywhere except at the vertices σ^0, where the sum of the dihedral angles, $\theta(\sigma^2)$, of the incident triangles σ^2's is in excess (negative curvature) or in defect (positive curvature) with respect to the 2π flatness constraint. The corresponding deficit angle ε is defined by $\varepsilon = 2\pi - \sum_{\sigma^2} \theta(\sigma^2)$, where the summation is extended to all 2 -dimensional simplices incident on the given σ^0. The automorphism group $Aut(P_{T_l})$ of $|P_{T_l}| \to M$, (i.e., the set of bijective maps preserving the incidence relations defining the polytopal structure), is the automorphism group of the edge refinement Γ (see [9]) of the 1-skeleton of the conical Regge polytope $|P_{T_l}| \to M$. Such a Γ is the 3-valent graph

$$\Gamma = \left(\{\rho^0(h,j,k)\} \bigsqcup^{N_1(T)} \{W(h,j)\}, \{\rho^1(h,j)^+\} \bigsqcup^{N_1(T)} \{\rho^1(h,j)^-\} \right). \tag{2.1}$$

where the vertex set $\{\rho^0(h,j,k)\}^{N_2(T)}$ is identified with the barycenters of the triangles $\{\sigma^o(h,j,k)\}^{N_2(T)} \in |T_l| \to M$, whereas each edge $\rho^1(h,j) \in \{\rho^1(h,j)\}^{N_1(T)}$ is generated by two half-edges $\rho^1(h,j)^+$ and $\rho^1(h,j)^-$ joined through the barycenters $\{W(h,j)\}^{N_1(T)}$ of the edges $\{\sigma^1(h,j)\}$ belonging to the original triangulation $|T_l| \to M$. The (counterclockwise) orientation in the 2-cells $\{\rho^2(k)\}$ of $|P_{T_l}| \to M$ gives rise to a cyclic ordering on the set of half-edges $\{\rho^1(h,j)^\pm\}^{N_1(T)}$ incident on the vertices $\{\rho^0(h,j,k)\}^{N_2(T)}$. According to these remarks, the (edge-refinement of the) 1-skeleton of $|P_{T_l}| \to M$ is a ribbon (or fat) graph [9], *viz.*, a graph Γ together with a cyclic ordering on the set of half-edges incident to each vertex of Γ. Conversely, any ribbon graph Γ characterizes an oriented surface $M(\Gamma)$ with boundary possessing Γ as a spine, (*i.e.*, the inclusion $\Gamma \hookrightarrow M(\Gamma)$ is an homotopy equivalence). In this way (the edge-refinement of) the 1-skeleton of a generalized conical Regge polytope $|P_{T_l}| \to M$ is in a one-to-one correspondence with trivalent metric ribbon graphs.

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It is possible to naturally relax the singular Euclidean structure associated with the conical polytope $|P_{T_l}| \to M$ to a complex structure $((M; N_0), \mathcal{C})$. Such a relaxing is defined by exploiting [9] the ribbon graph Γ (see (2.1)). Explicitly, let $\rho^2(h)$, $\rho^2(j)$, and $\rho^2(k)$ respectively be the two-cells $\in |P_{T_l}| \to M$ barycentrically dual to the vertices $\sigma^0(h)$, $\sigma^0(j)$, and $\sigma^0(k)$ of a triangle $\sigma^2(h,j,k) \in |T_l| \to M$. Let us denote by $\rho^1(h,j)$ and $\rho^1(j,h)$, respectively, the oriented edges of $\rho^2(h)$ and $\rho^2(j)$ defined by

$$\rho^1(h,j) \bigsqcup \rho^1(j,h) \doteq \partial\rho^2(h) \bigcap_{\Gamma} \partial\rho^2(j), \tag{2.2}$$

i.e., the portion of the oriented boundary of Γ intercepted by the two adjacent oriented cells $\rho^2(h)$ and $\rho^2(j)$ (thus $\rho^1(h,j) \in \rho^2(h)$ and $\rho^1(j,h) \in \rho^2(j)$ carry opposite orientations). Similarly, we shall denote by $\rho^0(h,j,k)$ the 3-valent, cyclically ordered, vertex of Γ defined by

$$\rho^0(h,j,k) \doteq \partial\rho^2(h) \bigcap_{\Gamma} \partial\rho^2(j) \bigcap_{\Gamma} \partial\rho^2(k). \tag{2.3}$$

To the edge $\rho^1(h,j)$ of $\rho^2(h)$ we associate [9] a complex coordinate $z(h,j)$ defined in the strip

$$U_{\rho^1(h,j)} \doteq \{z(h,j) \in \mathbb{C} | 0 < Re z(h,j) < L(h,j)\}, \tag{2.4}$$

$L(h,j)$ being the length of the edge considered. The coordinate $w(h,j,k)$, corresponding to the 3-valent vertex $\rho^0(h,j,k) \in \rho^2(h)$, is defined in the open set

$$U_{\rho^0(h,j,k)} \doteq \{w(h,j,k) \in \mathbb{C} \mid |w(h,j,k)| < \delta,\ w(h,j,k)[\rho^0(h,j,k)] = 0\}, \tag{2.5}$$

where $\delta > 0$ is a suitably small constant. Finally, the generic two-cell $\rho^2(k)$ is parametrized in the unit disk

$$U_{\rho^2(k)} \doteq \{\zeta(k) \in \mathbb{C} \mid |\zeta(k)| < 1,\ \zeta(k)[\sigma^0(k)] = 0\}, \tag{2.6}$$

where $\sigma^0(k)$ is the vertex $\in |T_l| \to M$ corresponding to the given two-cell.
We define the complex structure $((M; N_0), \mathcal{C})$ by coherently gluing, along the pattern associated with the ribbon graph Γ, the local coordinate neighborhoods $\{U_{\rho^0(h,j,k)}\}_{(h,j,k)}^{N_2(T)}$, $\{U_{\rho^1(h,j)}\}_{(h,j)}^{N_1(T)}$, and $\{U_{\rho^2(k)}\}_{(k)}^{N_0(T)}$. Explicitly, let $\{U_{\rho^1(h,j)}\}$, $\{U_{\rho^1(j,k)}\}$, $\{U_{\rho^1(k,h)}\}$ be the three generic open strips associated with the three cyclically oriented edges $\rho^1(h,j)$, $\rho^1(j,k)$, $\rho^1(k,h)$ incident on the vertex $\rho^0(h,j,k)$. Then the corresponding coordinates $z(h,j)$, $z(j,k)$, and $z(k,h)$ are related to $w(h,j,k)$ by the transition functions

$$w(h,j,k) = \begin{cases} z(h,j)^{\frac{2}{3}}, \\ e^{\frac{2\pi}{3}\sqrt{-1}} z(j,k)^{\frac{2}{3}}, \\ e^{\frac{4\pi}{3}\sqrt{-1}} z(k,h)^{\frac{2}{3}}, \end{cases} . \tag{2.7}$$

Similarly, if $\{U_{\rho^1(h,j_\beta)}\}$, $\beta = 1, 2, \ldots, q(k)$ are the open strips associated with the $q(k)$ (oriented) edges $\{\rho^1(h,j_\beta)\}$ boundary of the generic polygonal cell $\rho^2(h)$, then the transition functions between the corresponding coordinate $\zeta(h)$ and each $\{z(h,j_\beta)\}$ are given by [9]

$$\zeta(h) = \exp\left(\frac{2\pi\sqrt{-1}}{L(h)}\left(\sum_{\beta=1}^{\nu-1} L(h,j_\beta) + z(h,j_\nu)\right)\right), \qquad \nu = 1, \ldots, q(h), \tag{2.8}$$

with $\sum_{\beta=1}^{\nu-1} \cdot \doteq 0$, for $\nu = 1$, and where $L(h)$ denotes the perimeter of $\partial(\rho^2(h))$. Iterating such a construction for each vertex $\{\rho^0(h,j,k)\}$ in the conical polytope $|P_{T_l}| \to M$ we get a very explicit characterization of $((M;N_0),\mathcal{C})$.

Such a construction has a natural converse which allows us to describe the conical Regge polytope $|P_{T_l}| \to M$ as a uniformization of $((M;N_0),\mathcal{C})$. In this connection, the basic observation is that, in the complex coordinates introduced above, the ribbon graph Γ naturally corresponds to a Jenkins-Strebel quadratic differential ϕ with a canonical local structure which is given by [9]

$$\phi \doteq \begin{cases} \phi(h)|_{\rho^1(h)} = dz(h) \otimes dz(h), \\ \\ \phi(j)|_{\rho^0(j)} = \frac{9}{4} w(j) dw(j) \otimes dw(j), \\ \\ \phi(k)|_{\rho^2(k)} = -\frac{[L(k)]^2}{4\pi^2 \zeta^2(k)} d\zeta(k) \otimes d\zeta(k), \end{cases} \tag{2.9}$$

where $L(k)$ denotes the perimeter of $\partial(\rho^2(k))$, and where $\rho^0(h,j,k)$, $\rho^1(h,j)$, $\rho^2(k)$ run over the set of vertices, edges, and 2-cells of $|P_{T_l}| \to M$. If we denote by

$$\Delta_k^* \doteq \{\zeta(k) \in \mathbb{C} |\ 0 < |\zeta(k)| < 1\}, \tag{2.10}$$

the punctured disk $\Delta_k^* \subset U_{\rho^2(k)}$, then for each given deficit angle $\varepsilon(k) = 2\pi - \theta(k)$ we can introduce on each Δ_k^* the conical metric

$$ds^2_{(k)} \doteq \frac{[L(k)]^2}{4\pi^2} |\zeta(k)|^{-2\left(\frac{\varepsilon(k)}{2\pi}\right)} |d\zeta(k)|^2 = |\zeta(k)|^{2\left(\frac{\theta(k)}{2\pi}\right)} |\phi(k)_{\rho^2(k)}|, \tag{2.11}$$

where

$$|\phi(k)_{\rho^2(k)}| = \frac{[L(k)]^2}{4\pi^2 |\zeta(k)|^2} |d\zeta(k)|^2. \tag{2.12}$$

is the standard cylindrical metric associated with the quadratic differential $\phi(k)_{\rho^2(k)}$. Thus, the punctured Riemann surface $((M;N_0),\mathcal{C})$ associated with the conical Regge polytope $|P_{T_l}| \to M$ is provided by

$$((M;N_0),\mathcal{C});\{ds^2_{(k)}\}) = \bigcup_{\{\rho^0(h,j,k)\}}^{N_2(T)} U_{\rho^0(h,j,k)} \bigcup_{\{\rho^1(h,j)\}}^{N_1(T)} U_{\rho^1(h,j)} \bigcup_{\{\rho^2(k)\}}^{N_0(T)} (\Delta_k^*, ds^2_{(k)}).$$

Although the above correspondence between conical Regge polytopes and punctured Riemann surfaces is rather natural there is yet another uniformization representation of $|P_{T_l}| \to M$ which is of relevance while discussing conformal field theory on a given $|P_l| \to M$. The point is that the analysis of a CFT on a singular surface such as $|P_{T_l}| \to M$ calls for the imposition of suitable boundary conditions in order to take into account the conical singularities of the underlying Riemann surface $((M;N_0),\mathcal{C}, ds^2_{(k)})$. This is a rather delicate issue since conical metrics give rise to difficult technical problems in discussing the glueing properties of the resulting conformal fields. In boundary conformal field theory, problems of this sort are taken care of by tacitly assuming that a neighborhood of the possible boundaries is endowed with a cylindrical metric. In our setting such a prescription naturally calls into play the metric associated with the quadratic differential ϕ, and requires that we regularize into finite cylindrical ends the cones $(\Delta_k^*, ds^2_{(k)})$. Such a regularization is realized by noticing that if we introduce the annulus

$$\Delta^*_{\theta(k)} \doteq \left\{\zeta(k) \in \mathbb{C} | e^{-\frac{2\pi}{\theta(k)}} \leq |\zeta(k)| \leq 1\right\} \subset \overline{U_{\rho^2(k)}}, \tag{2.13}$$

then the surface with boundary

$$M_\partial \doteq ((M_\partial; N_0), \mathcal{C}) = \bigcup U_{\rho^0(j)} \bigcup U_{\rho^1(h)} \bigcup (\Delta^*_{\theta(k)}, \phi(k)) \tag{2.14}$$

defines the blowing up of the conical geometry of $((M; N_0), \mathcal{C}, ds^2_{(k)})$ along the ribbon graph Γ.

The metrical geometry of $(\Delta^*_{\theta(k)}, \phi(k))$ is that of a flat cylinder with a circumference of length given by $L(k)$ and height given by $L(k)/\theta(k)$, (this latter being the slant radius of the generalized Euclidean cone $(\Delta_k^*, ds^2_{(k)})$ of base circumference $L(k)$ and vertex conical angle $\theta(k)$).We also have

$$\begin{aligned} \partial M_\partial &= \bigsqcup_{k=1}^{N_0} S^{(+)}_{\theta(k)}, \\ \partial \Gamma &= \bigsqcup_{k=1}^{N_0} S^{(-)}_{\theta(k)} \end{aligned} \tag{2.15}$$

where the circles

$$\begin{aligned} S^{(+)}_{\theta(k)} &\doteq \left\{\zeta(k) \in \mathbb{C} || \zeta(k)| = e^{-\frac{2\pi}{\theta(k)}}\right\}, \\ S^{(-)}_{\theta(k)} &\doteq \{\zeta(k) \in \mathbb{C} || \zeta(k)| = 1\} \end{aligned} \tag{2.16}$$

respectively denote the inner and the outer boundary of the annulus $\Delta^*_{\theta(k)}$. Note that by collapsing $S^{(+)}_{\theta(k)}$ to a point we get back the original cones $(\Delta_k^*, ds^2_{(k)})$. Thus, the surface with boundary M_∂ naturally corresponds to the ribbon graph Γ associated with the 1-skeleton $K_1(|P_{T_l}| \to M)$ of the polytope $|P_{T_l}| \to M$, decorated with the finite cylinders $\{\Delta^*_{\theta(k)}, |\phi(k)|\}$. In such a framework the conical angles $\{\theta(k) = 2\pi - \varepsilon(k)\}$ appears as (reciprocal of) the moduli m_k of the annuli $\{\Delta^*_{\theta(k)}\}$,

$$m(k) = \frac{1}{2\pi} \ln \frac{1}{e^{-\frac{2\pi}{\theta(k)}}} = \frac{1}{\theta(k)} \tag{2.17}$$

(recall that the modulus of an annulus $r_0 < |\zeta| < r_1$ is defined by $\frac{1}{2\pi} \ln \frac{r_1}{r_0}$). According to these remarks we can equivalently represent the conical Regge polytope $|P_{T_l}| \to M$ with the uniformization $((M; N_0), \mathcal{C}); \{ds^2_{(k)}\})$ or with its blowed up version M_∂.

In order to exploit the above geometrical set up in the study of open/closed string dualities, let us consider D real scalar maps $X^\alpha : M_\partial \to \mathcal{T}$, $i = 1, \ldots, D-1$, injecting M_∂ into an unspecified target space $\mathcal{T}$ and let us first focus on a fixed, but otherwise generic,

$\Delta^*_{\varepsilon(p)}$. Although quantization of (non-critical) Polyakov string on a annular domain is an overkilled topic, it is worthwhile discussing it in some detail, both to fix notation and to deal with some of the subtleties arising from to the combinatorial origin of M_∂.

The world-sheet action is:

$$S = \frac{1}{4\pi}\int d\zeta(p)d\bar{\zeta}(p)\, G_{\alpha\beta}(p)\partial X^\alpha(p)\overline{\partial X}^\beta(p) + B_{\alpha\beta}(p)\partial X^\alpha(p)\overline{\partial X}^\beta(p) - \frac{1}{2}\Phi(p)\, R^{(2)}. \quad (2.18)$$

The geometry of target space is specified by a suitable assignation of the background matrix

$$E(p) = G(p) + B(p), \quad (2.19)$$

which encodes informations about the background metric $G_{\alpha\beta}(p)$ and the Kalb-Ramond field $B_{\alpha\beta}(p)$ components. $\Phi(p)$ is a properly chosen dilaton field. In particular, we will deal with flat toroidal backgrounds, i.e. we will consider a string moving in a background in which D dimensions are compactified whereas the metric, the Kalb-Ramond field and the dilaton are independent from the spacetime coordinates $X^\alpha, \alpha = 1, \ldots, D$.

Since in the description of the metric geometry of the triangulation $|T_l| \to M$ as the dual open Riemann surface M_∂ we are, roughly speaking, unwrapping conical 2-cells into finite cylindrical ends [10], we can adopt for the matter sector the most general condition:

$$X^\alpha(p)(e^{2\pi i}\zeta,\, e^{-2\pi i}\bar{\zeta}) = X^\alpha(p)(\zeta,\, \bar{\zeta}) + 2\nu^\alpha(p)\pi\frac{R^\alpha(p)}{l(p)}, \qquad \nu^\alpha(p) \in \mathbb{Z} \quad (2.20)$$

according to which each field $X^\alpha(p)$ winds $\nu^\alpha(p)$ times around the corresponding toroidal cycles of length $\frac{R^\alpha(p)}{l(p)}$ in the compact target space $\mathcal{T}$. Here $l(p)$ is a length parameter built out of the geometric assigned data of the original triangulation.

In this way, if we further put to zero the dilaton and B-field components, we are actually encoding all data about the background geometry in the value of the compactification radius, letting the metric to be diagonal and decoupling the model in each direction. Hence, we can consider just the quantization of a single scalar field. The world-sheet action on $\Delta^*_{\varepsilon(k)}$ becomes:

$$S = \frac{1}{8\pi}\int_{\Delta^*_{\epsilon(p)}} d\zeta(p)d\bar{\zeta}(p)\,\partial X(p)\,\overline{\partial X}(p). \quad (2.21)$$

The extension to a D-dimensional background will be straightforward from a target-space point of view.

Since the theories on the various cylindrical ends are effectively decoupled, from now on we shall suppress the polytope index (k) and we will restore it once we will describe the interaction of the distinct models along the ribbon graph Γ.

The fundamental prerequisite to quantize a Conformal Field Theory (CFT) on a surface with boundary is to have the full control of the same quantum theory on the entire complex plane, the latter being usually referred to as the *bulk theory.* This is defined via a suitable assignation of an Hilbert space of states $\mathcal{H}^{(C)}$, endowed with the action of an Hamiltonian operator $H^{(C)}$ and of a vertex operation, i.e. a formal map $\Phi^{(C)}(\circ;\zeta,\bar{\zeta}) : \mathcal{H}^{(C)} \to \mathrm{End}\left[V[\zeta,\bar{\zeta}]\right]$ associating to each vector $|\phi\rangle \in \mathcal{H}^{(C)}$ a conformal field $\phi(\zeta,\bar{\zeta})$ of conformal dimension $h, \bar{h}$. The bulk theory is completely worked out once we know the coefficients of the Operator Product Expansion (OPE) for all fields in the theory. Actually, we can face this task for most CFTs since, among conformal fields, a preferential role is played by chiral ones, whose Laurent modes generate two isomorphic copies of the chiral algebra which defines the symmetries of the theory.

In our case such a role is played by chiral currents $J(z) = i\,\partial X(z) = \sum_n \mathfrak{a}_n\, z^{-n-1}$ and $\overline{J}(\bar{\zeta}) = i\,\overline{\partial X}(\bar{\zeta}) = \sum_n \overline{\mathfrak{a}}_n\, \bar{\zeta}^{-n-1}$ which generate two independent copies of the Heisenberg algebra:

$$[\mathfrak{a}_n\,,\,\mathfrak{a}_m] = n\delta_{n+m,0} \qquad [\overline{\mathfrak{a}}_n\,,\,\overline{\mathfrak{a}}_m] = n\delta_{n+m,0} \qquad [\mathfrak{a}_n\,,\,\overline{\mathfrak{a}}_m] = 0. \tag{2.22}$$

The Virasoro fields T and $\overline{T}$ play a special role among the chiral fields of a CFT. Their modes L_n and $\overline{L}_n$ close two copies of the Virasoro algebra:

$$\begin{aligned} [L_m\,,\,L_n] &= (m-n)\,L_{m+n} + \frac{1}{12}n(n^2-1)\delta_{m+n,0}, \\ \left[\overline{L}_m\,,\,\overline{L}_n\right] &= (m-n)\,\overline{L}_{m+n} + \frac{1}{12}n(n^2-1)\delta_{m+n,0}. \end{aligned}$$

Since the Virasoro algebra belongs to the universal covering of the Virasoro one, we can represent its generators by means of the Sugawara construction:

$$L_0 \doteq \sum_{n>0} \mathfrak{a}_{-n}\cdot\mathfrak{a}_n + \frac{1}{2}(\mathfrak{a}_0)^2, \qquad L_n \doteq \frac{1}{2}\sum_{m\in\mathbb{Z}} :\mathfrak{a}_{n-m}\cdot\mathfrak{a}_m:, \tag{2.23}$$

hence allowing for an immediate definition of the bulk Hamiltonian operator $H^{(C)}$:

$$H^{(C)} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{D}{12}\right).$$

Moreover, their action determines a diagonal decomposition of the Hilbert space into subspaces carrying irreducible representations of the two commuting chiral algebras:

$$\mathcal{H}^{(C)} \doteq \bigoplus_{\lambda\,\overline{\lambda}} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_{\overline{\lambda}}, \tag{2.24}$$

where $\lambda_{(\mu,\nu)} = \mu\frac{l}{R} + \frac{1}{2}\nu\frac{R}{l}$ and $\overline{\lambda}_{(\mu,\nu)} = \mu\frac{l}{R} - \frac{1}{2}\nu\frac{R}{l}$ are respectively the $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$ charges (real numbers).

2.1 Amplitude on $\Delta^*_{\varepsilon(p)}$

The bulk CFT's properties we briefly summarized in last section are the main ingredients to discuss in detail the extension of the same CFT on a given cylindrical end over M_∂. As a matter of fact, remembering that, from a microscopic point of view, to define a CFT on a surface with boundary means to work out which values we can consistently assign to fields on the boundaries of the new domain (i.e. which boundary conditions we can choose), the key datum we must keep track of to fulfill this goal are the OPE coefficients of the bulk theory. The latter identify an algebra of fields the boundary assignations must be compatible with. Thus, the recipe we will follow aims, firstly, to look for all possible

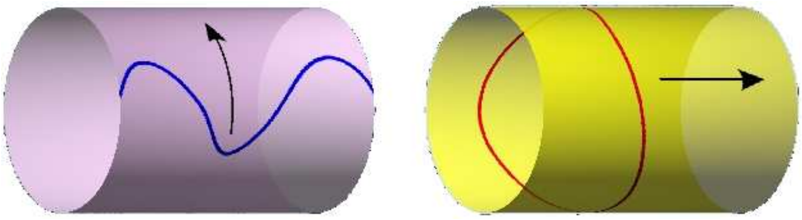

Figure 1: Dual cylinders.

boundary assignations. We will show that these can be encoded into a set of coherent boundary states which arise as a generalization of those introduced in [15]. Secondly, we will consider the list of constraints which the bulk algebra of fields induces on the boundary components to select, among the above set of boundary assignations, those compatible with the algebra itself.

Thus, let us consider any but fixed cylindrical end $\Delta^*_{\varepsilon(k)}$. In a string theory perspective, it can be viewed both as an open string one-loop diagram or as the tree level diagram of a closed string propagating for a finite length path. In the first (direct channel) picture, time flows around the cylinder and the associated quantization scheme defines functions of the modular parameter $\tau(p) = i\theta(p) = i(2\pi - \varepsilon(p))$. On the opposite, in the second (transverse channel) framework, time flows along the cylinder, and the associated quantization scheme is related to the former by means of the modular transformation $\tau(p) \to -\frac{1}{\tau(p)}$. In the forthcoming analysis, we will switch back and forth between these two points of view.

The key object we wish to calculate is the amplitude associated to this diagram; this is a deep-investigated problem whenever the boundary assignations a priori satisfy the canonical prescription of Neumann or Dirichlet conditions [16]. However, within the discretized model we are dealing with, cylindrical ends arise as a byproduct of an unwrapping process of a conical structure [10, 12]. Hence we do not have a priori a full control on the behavior of matter fields on the vertexes of the parent triangulation when these spread over the full outer cylinder boundary[1] $S^{(-)}_{\varepsilon(p)}$. Thus, in our description, we will follow the procedure outlined by Charpentier and Gawedzky in [17], which allows to write the amplitude on an arbitrary Riemann surface Σ with a fixed number of boundary loops S_I, parametrized by analytical real maps $p_I : S^1 \to S_I$, and by an arbitrary specification of matter fields on them.

Within this framework we get:

$$\mathcal{A}_\Sigma = \int_{\{X \circ p_I = X_I\}} \mathcal{D}[X]\, e^{-S[X]}, \tag{2.25}$$

where $S[X]$ is the Euclidean action of the bulk CFT, $\mathcal{D}[X]$ is the formal measure on the target space and the kinematical configurations of the field X are those such that it assumes the general but fixed value X_I over the given boundary loop S_I.

[1] Within this manuscript, we keep the convention of [10, 12] to refer to $S^{(+)}_{\varepsilon(p)}$ as the boundary of the cylinder glued to the ribbon graph (inner boundary in the annuli picture) whereas $S^{(-)}_{\varepsilon(p)}$ is the "free boundary" (outer boundary in the annuli picture).

This rather abstract and formal expression acquires a precise meaning when we deal with real scalar fields defined as injection maps from Σ to a flat toroidal background. In this case, it is always possible to decompose $X = X_{\rm cl} + \tilde{X}$ where the real map $X_{\rm cl}$ is an harmonic function w.r.t. Δ_Σ (the Laplacian operator defined over Σ) fulfilling the boundary assignation (i.e. $X_{\rm cl} \circ p_I = X_I$). $\tilde{X} : \Sigma \to \mathbb{R}$ is the collection of the off-shell modes of X satisfying $\tilde{X} \circ p_I = 0$. This constraint implies the diagonal decomposition of the bulk action, $S[X] = S[X_{\rm cl}] + S[\tilde{X}]$. If we specify $\Sigma \doteq \Delta^*_{\varepsilon(k)}$, we get:

$$\mathcal{A}_{\Delta^*_{\varepsilon(p)}} = \frac{1}{4\pi} \frac{1}{\eta(\tilde{q})} \sum_{X_{\rm cl}} e^{-S[X_{\rm cl}]}, \tag{2.26}$$

where $\eta(\tilde{q})$ is the Dedekind-η function with $\tilde{q} = e^{-\frac{2\pi i}{\tau(p)}}$ and where the sum runs over the set of classical solutions.

According to the prescription introduced in [17] and specialized to the compactified boson in [15], it is possible to parametrize the classical field (zero mode) in term of its restrictions to the boundaries $S^{(+)}_{\varepsilon(p)}$ and $S^{(-)}_{\varepsilon(p)}$. As a byproduct, the space of classical solutions is fully parametrized by the two set of complex numbers $\{a_n\}$ and $\{b_n\}$, obeying the reality conditions $a_{-n} = \overline{a}_n$ and $b_{-n} = \overline{b}_n$, a real number $t \in \left(0,\, 2\pi \frac{R}{l}\right]$ and a pair of integers $(\mu, \nu) \in \mathbb{Z}^2$. The latter are in one-to-one relation with the two integers parametrizing the irreducible representations of the chiral algebra (see eq. (2.24) and comments below). The existence of such 1:1 correspondence can be fully exploited to make explicit the formal map between admissible boundary conditions and coherent states built out of linear combinations of elements in the bulk Fock space. In details, splitting $t = t_- - t_+$, we map each boundary assignation labelled by $\{(\mu, \nu),\, \{a_n\},\, t_-\}$ (resp. $\{(\mu, \nu),\, \{b_n\},\, t_+\}$) into $\left|\mathfrak{r}^\alpha_{(\mu,\nu)}(S^{(-)}_{\varepsilon(k)})\right\rangle$ (resp. $\left|\mathfrak{r}^\alpha_{(\mu,\nu)}(S^{(+)}_{\varepsilon(k)})\right\rangle) \in \mathcal{H}_{\mu,\nu} \otimes \overline{\mathcal{H}}_{\mu,\nu} \subset \mathcal{H}^{(C)}$.

Therefore, the amplitude on the fixed cylindrical end can be written as

$$\mathcal{A}(\{a_n\}\, \{b_n\}\, , t) = \sum_{(\mu,\nu)} \left\langle \mathfrak{r}_{(\mu,\nu)}(S^{(+)}_{\varepsilon(p)}) \right| \tilde{q}^{L_0 + \bar{L}_0 - \frac{c}{12}} \left| \mathfrak{r}_{(\mu,\nu)}(S^{(-)}_{\varepsilon(p)}) \right\rangle. \tag{2.27}$$

These boundary states are the following generalization of those introduced in [15]:

$$\begin{aligned}\left|\mathfrak{r}_{(\mu,\nu)}(S^{(-)}_{\varepsilon(p)})\right\rangle &= e^{it_-(\lambda_{(\mu,\nu)} + \overline{\lambda}_{(\mu,\nu)})} \times \\ &\times \prod_{n=1}^{\infty} \sum_{m_1,m_2} A^n_{m_1,m_2}(a_n, a_{-n}) \frac{(\mathfrak{a}_{-n})^{m_1} \otimes (\overline{\mathfrak{a}}_{-n})^{m_2}}{\sqrt{n^{m_1+m_2} m_1! m_2!}} \left|(\mu,\nu)\right\rangle, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\left|\mathfrak{r}_{(\mu,\nu)}(S^{(+)}_{\varepsilon(p)})\right\rangle &= e^{it_+(\lambda_{(\mu,\nu)} + \overline{\lambda}_{(\mu,\nu)})} \times \\ &\times \prod_{n=1}^{\infty} \sum_{m_1,m_2} B^n_{m_1,m_2}(b_n, b_{-n}) \frac{(\mathfrak{a}_{-n})^{m_1} \otimes (\overline{\mathfrak{a}}_{-n})^{m_2}}{\sqrt{n^{m_1+m_2} m_1! m_2!}} \left|(\mu,\nu)\right\rangle, \end{aligned} \tag{2.29}$$

with

$$A^n_{m_1,m_2}(a_n, a_{-n}) = e^{i\pi s(m_1+m_2)} \times \tag{2.30}$$
$$\begin{cases} (2i\sqrt{n}a_n)^{m_1-m_2}\sqrt{\frac{m_2!}{m_1!}}e^{-2n|a_n|^2}L^{(m_1-m_2)}_{m_2}(4n|a_n|^2), & m_1 \geq m_2 \\ (2i\sqrt{n}a_n)^{m_2-m_1}\sqrt{\frac{m_1!}{m_2!}}e^{-2n|a_n|^2}L^{(m_2-m_1)}_{m_1}(4n|a_n|^2), & m_2 \geq m_1 \end{cases}$$

being $s \in \mathbb{R}$ and $L^{(m_1-m_2)}_{m_2}(\circ)$ the m_2-th Laguerre polynomial of the $(m_1 - m_2)$-th kind. Replacing a_n with b_n and acting by conjugation (induced by the orientation of the boundary), we end up with $B^n_{m_1,m_2}(b_n, b_{-n}) = \overline{A^n_{m_1,m_2}(b_{-n}, b_n)}$. We leave a reader interested in the precise derivation to [18].

Although exhaustive from a mathematical point of view, as anticipated at the beginning of the section the answer we reached with (2.28) and (2.29) is not yet conclusive. As a matter of fact, from a physical point of view, the presence of a boundary allows us to rephrase the whole process macroscopically considering the presence of two branes which, in the transverse channel, emit and absorb a closed string (whose initial and final states are described by the above boundary states) via non-perturbative processes, while, in the direct channel they are the objects where the endpoints of the open string running one loop lay on. In this connection, BCFT is the natural mean to describe microscopically the brane-string bound state, without any reference to spacetime geometry. As a consequence, we need to avoid any information flow through the boundary itself (the cylinder or the annulus boundary in our setting) and, to this avail, chiral and Virasoro fields must satisfy appropriate glueing conditions along it. In particular, the holomorphic and antiholomorphic components of the latter must coincide on the annulus boundary:

$$\zeta^2\, T(\zeta)_{|\zeta|=e^{\frac{2\pi}{2\pi-\varepsilon(p)}}} = \bar{\zeta}^2\bar{T}(\bar{\zeta})|_{|\zeta|=e^{\frac{2\pi}{2\pi-\varepsilon(p)}}} \quad \text{and} \quad \zeta^2\, T(\zeta)_{|\zeta|=1} = \bar{\zeta}^2\bar{T}(\bar{\zeta})|_{|\zeta|=1}. \tag{2.31a}$$

The analogue condition on the chiral currents is weaker; they must be related by an automorphism Ω of the chiral algebra:

$$\zeta\, J(\zeta)_{|\zeta|=e^{\frac{2\pi}{2\pi-\varepsilon(p)}}} = \bar{\zeta}\Omega\bar{J}(\bar{\zeta})|_{|\zeta|=e^{\frac{2\pi}{2\pi-\varepsilon(p)}}} \quad \text{and} \quad \zeta\, J(\zeta)_{|\zeta|=1} = \bar{\zeta}\Omega\bar{J}(\bar{\zeta})|_{|\zeta|=1}. \tag{2.31b}$$

Being the $\mathfrak{u}(1)$ algebra Abelian, its automorphism group is $\mathbb{Z}_2$, thus $\Omega = \pm 1$. Exploiting radial quantization the above glueing conditions translate into projection maps acting on boundary states:

$$\left(L_n - \overline{L}_{-n}\right) \|B\rangle\!\rangle = 0, \tag{2.32}$$

and

$$(\mathfrak{a}_n + \bar{\mathfrak{a}}_{-n}) \|B\rangle\!\rangle = 0, \qquad \text{if } \Omega = -1 \tag{2.33a}$$
$$(\mathfrak{a}_n - \bar{\mathfrak{a}}_{-n}) \|B\rangle\!\rangle = 0. \qquad \text{if } \Omega = +1. \tag{2.33b}$$

The Sugawara construction ensures that conditions (2.33a) and (2.33b) are sufficient to enforce conformal invariance encoded in (2.32). Their application projects (2.28) and (2.29) into the ordinary Neumann and Dirichlet boundary states defined for the compactified bosonic field $X(\zeta, \bar{\zeta})$:

$$\left|\mathfrak{r}_{(\mu,\nu)}(S^{(-)}_{\varepsilon(p)})\right\rangle^{(D)} = \frac{1}{\sqrt{2\frac{L(p)}{R(p)}}} \sum_{\mu\in\mathbb{Z}} e^{it_+\mu\frac{L(p)}{R(p)}} e^{\sum_{n=1}^{\infty}\frac{1}{n}\mathfrak{a}_{-n}\bar{\mathfrak{a}}_{-n}} \left|(\mu,0)\right\rangle , \tag{2.34a}$$

$$\left|\mathfrak{r}_{(\mu,\nu)}(S^{(-)}_{\varepsilon(p)})\right\rangle^{(N)} = \sqrt{\frac{L(p)}{R(p)}} \sum_{\nu\in\mathbb{Z}} e^{i\tilde{t}_+\frac{\nu}{2}\frac{R(p)}{L(p)}} e^{-\sum_{n=1}^{\infty}\frac{1}{n}\mathfrak{a}_{-n}\bar{\mathfrak{a}}_{-n}} \left|(0,\nu)\right\rangle . \tag{2.34b}$$

An equivalent relation clearly holds for $\left|\mathfrak{r}_{(\mu,\nu)}(S^{(+)}_{\varepsilon(p)})\right\rangle$. The careful demonstration of the above statements, which exploits recursion relations of the Laguerre polynomials, is reported in [18], section 2.3 and appendix A.

2.2 Interactions on Γ: boundary insertion operators

With the previous analysis we determined the set of boundary states representing the admissible field assignations over each $\Delta^*_{\varepsilon(k)}$ boundary component. This not only completes the first of the two step programme outlined at the beginning of the section, but it is instrumental for the next one, the discussion on the interaction along the ribbon graph Γ among the N_0 distinct copies of the theory, each one living on a different cylindrical end.

The existence of pairwise adjacent boundary conditions led us to propose in [11] that this interaction could be mediated by boundary conditions changing operators, whose presence is predicted in the abstract formulation of boundary conformal field theory [19, 20]. As a matter of fact, in the standard scenario of a BCFT defined on the the Upper Half Plane, the prescribed boundary condition can change along the real axis. In a radial quantization scheme, such a situation is explained with the presence of a vacuum which is no longer invariant under the action of the Virasoro operator $L^{(H)}_{-1}$. In [19] it was proposed that such states are obtained by the local action of a specific operator acting on the true vacuum and supported only on the boundary, i.e. it induces a transition between boundary conditions. According to the vertex operation, each of these operators can be associated to a specific vector in the Fock space dependent upon boundary data and such that it cannot be correlated with bulk fields by means of a bulk to boundary OPE.

However, the described local action of a boundary condition changing operator does not fit in our discretized model. As a matter of fact, in the framework dual to a Random Regge Triangulation, the N_0 cylinders are pairwise glued together along one of their two boundaries (commonly the inner one in the annuli picture) through one ribbon graph edge. Hence, in this case, we should more properly speak of a "separation edge"[2] between two adjacent cylindrical ends. Furthermore, we do not have a jump between two boundary conditions taking place at a precise point. On the opposite, two different boundary conditions coexist in the adjacency limit along the whole edge [11], as depicted in figure 2.

Switching back to field theoretical contents, in this connection it is no longer correct to claim the presence of a vacuum state invariant under translations along the boundary. As

[2]In order to keep terminology and notations "under control", we shall often refer to the edge $\rho^1(p,q)$ in common between $\Delta^*_{\varepsilon(p)}$ and $\Delta^*_{\varepsilon(q)}$ as a boundary. We feel that the overall context allows the reader to point out which is the specific scenario we are dealing with.

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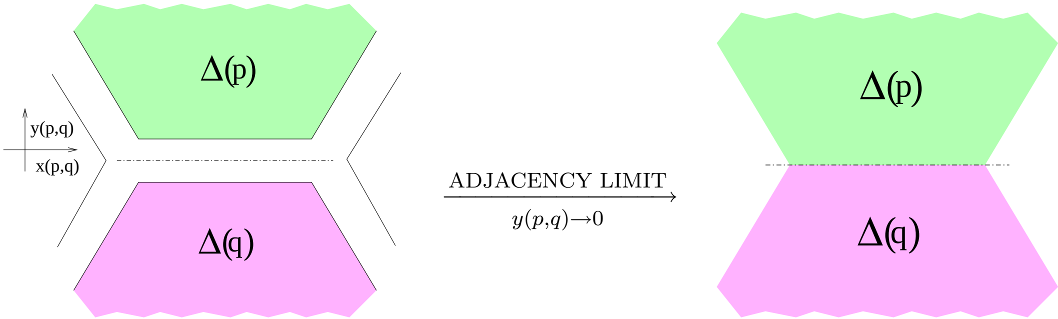

Figure 2: Shared boundaries in the adjacency limit.

a matter of facts, being the shared boundary obtained out of two separate loops, each of them being part of a domain where a BCFT is constructed, all the associated Fock space elements are invariant under translation only along the relevant boundary loop. Thus, in order for the geometric glueing process to be consistent with the functional data of the theory on each cylinder, we must require that the N_0 a priori independent Fock spaces blend pairwise without breaking the conformal and the chiral symmetry of the model. As we will show in the forthcoming discussion, this leads to the introduction of an additional class of operators which live on the boundary shared between two adjacent polytopes, carry an irreducible action of the chiral algebra and dynamically mediate between two adjacent boundary conditions.

To provide the details, let us consider two adjacent cylindrical ends $\Delta^*_{\varepsilon(p)}$ and $\Delta^*_{\varepsilon(q)}$; these ends are glued to the oriented boundaries $\partial\Gamma_p$ and $\partial\Gamma_q$ of the ribbon graph. Let us consider the oriented strip associated with the edge $\rho^1(p,q)$ of the ribbon graph and its uniformized neighborhood $\left(U_{\rho^1(p,q)}, z(p,q)\right)$, where the uniformizing coordinate $z(p,q)$ was defined as in (2.8). In this geometric background, let us focus on an unspecified BCFT on $\Delta^*_{\varepsilon(p)}$ and let us fix some notations:

- $W(\zeta)$ and $\overline{W}(\bar{\zeta})$ are respectively the set of holomorphic and antiholomorphic chiral fields of the parent bulk theory defining two commuting copies of the chiral algebra describing the symmetries of the model;
- $\mathcal{Y} = \{\lambda(p)\}$ is the collection of indexes labelling the irreducible representations of the chiral algebra associated to the BCFT on $\Delta^*_{\varepsilon(p)}$;
- $\mathcal{A} = \{A(p)\}$ is the set of possible boundary conditions we can assign on each boundary components, hence located at $|\zeta(p)| = 1$ and at $|\zeta(p)| = e^{\frac{2\pi}{2\pi-\varepsilon(p)}}$ in the annuli picture. Each $A(p)$ includes either the glueing automorphism, denoted as $\Omega_{A(p)}$, either a specification for all other necessary parameters (i.e., when dealing with the compactified boson, the brane position or the Wilson line).

Beside avoiding information flow through the boundary (see comments before formulae (2.31)), the existence of the glueing automorphism $\Omega_{A(p)}$ cited above gives rise to the action of a single copy of the chiral algebra on the state space $\mathcal{H}^{(O)}$ of the boundary theory. As a matter of fact, being defined only on a part of the full complex plane i.e. the annulus, $W(\zeta)$ and $\bar{W}(\bar{\zeta})$ are not sufficient to generate two copies of the chiral algebra. However, since the glueing condition $\zeta(p)^{h_W} W(\zeta(p))_{|\zeta(p)|=1} = \Omega_{A(p)} \zeta(p)^{\bar{h}_{\overline{W}}} \overline{W}(\bar{\zeta}(p))|_{|\zeta(p)|=1}$ (being h_W the conformal weight of $W(\zeta(p))$ states that holomorphic and antiholomorphic chiral fields are related on the boundary, we may introduce:[3]

$$\mathbf{W}_{\Omega_{A(p)}} = \begin{cases} W(\zeta(p)) & |\zeta(p)| \leq 1 \\ \Omega_{A(p)} \overline{W}(\bar{\zeta}(p)) & |\zeta(p)| > 1 \end{cases}, \tag{2.35}$$

which is a single analytic function on $\mathbb{C}$. Its Laurent expansion coherently defines a single copy $\mathcal{W}$ of the chiral algebra associated to the boundary conformal field theory on $\Delta^*_{\varepsilon(p)}$ [19, 20]. Hence it induces a decomposition of the *open* CFT Fock space $\mathcal{H}^{(O)}$ into a sum of carriers of its irreducible representations [21]: $\mathcal{H}^{(O)} = \bigoplus_\lambda \mathcal{H}_\lambda$, being $\mathcal{H}_\lambda$ the subspace appearing in (2.24).

The above construction and discussion holds for the BCFT defined on each cylinder. Suppose now to held fixed $\Delta^*_{\varepsilon(p)}$ and let us consider its adjacent cylinder $\Delta^*_{\varepsilon(q)}$. Referring to $B(q)$ as the boundary condition on its inner boundary out of the automorphism $\Omega_{B(q)}$, the glueing condition reads

$$\zeta(q)^{h_W} W(\zeta(q))_{|\zeta(q)|=1} = \Omega_{B(q)} \bar{\zeta}(q)^{\bar{h}_{\overline{W}}} \overline{W}(\bar{\zeta}(q))|_{|\zeta(q)|=1},$$

whereas the single chiral field is

$$\mathbf{W}_{\Omega_{B(q)}} = \begin{cases} W(\zeta(q)) & |\zeta(q)| \leq 1 \\ \Omega_{B(q)} \overline{W}(\bar{\zeta}(q)) & |\zeta(q)| > 1 \end{cases},$$

which is analytic on the full complex plane and whose Laurent modes define a single copy of the chiral algebra.

Within this framework we can implement a non symmetry-breaking glueing of two adjacent cylindrical ends associating to such a pair a unique copy of the chiral algebras and, by means of the Sugawara construction, of the Virasoro ones. To this avail, as a starting point we exploit (2.8) to express the holomorphic and the antiholomorphic components of the chiral fields defined on each cylindrical end in term of the strip coordinate, namely $\mathbf{W}_{\Omega_{A(p)}}(z(p,q)) = \mathbf{W}_{\Omega_{A(p)}}(\zeta(p)) \left(\frac{d\,z(p,q)}{d\,\zeta(p)}\right)^{-h_\mathbf{W}}$ and $\mathbf{W}_{\Omega_{B(q)}}(z(q,p)) = \mathbf{W}_{\Omega_{B(q)}}(\zeta(q)) \left(\frac{d\,z(q,p)}{d\,\zeta(q)}\right)^{-h_\mathbf{W}}$.

Taking into account $z(q,p) = -z(p,q)$, we perform the glueing requiring a condition similar to (2.31) to hold. In this process, the subtle point resides in the map Ω in (2.35). As a matter of fact, we must take into account that the whole process must relate the two glueing automorphisms $\Omega_{A(p)}$ and $\Omega_{B(q)}$ associated to the BCFTs defined respectively on $\Delta^*_{\varepsilon(p)}$ and $\Delta^*_{\varepsilon(q)}$. Thus it seems natural to introduce a further automorphism $\Omega'^{A(p)B(q)}$ which, in the adjacency limit $y(p,q) = \Im\left[z(p,q)\right] \to 0$, acts along the boundary deforming

[3] The reader should keep track of the following change of perspective: $\zeta(p)$ and $\bar{\zeta}(p)$ are no more independent coordinates but, in formulas such as (2.35) they are related by complex conjugation.

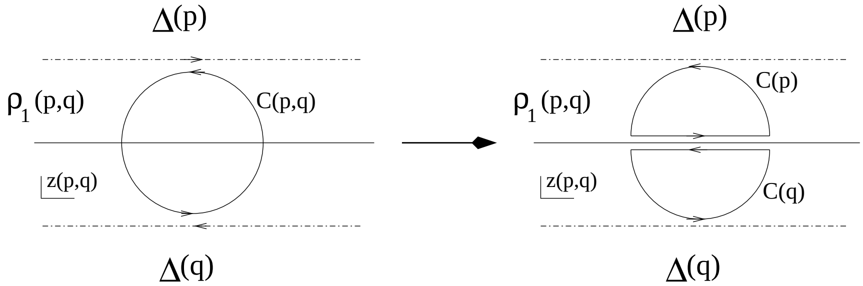

Figure 3: A small integration contour intersecting the (p,q) edge of the ribbon graph.

continuously the (holomorphic and antiholomorphic components of the) bulk chiral fields in $\Delta^*_{\varepsilon(p)}$ in the corresponding on $\Delta^*_{\varepsilon(q)}$. To rephrase:

$$\mathbf{W}_{\Omega_{A(p)}}(z(p,q))|_{y(p,q)\to 0} = \Omega'^{A(p)B(q)}\mathbf{W}_{\Omega_{B(q)}}(z(p,q))|_{y(p,q)\to 0}. \tag{2.36}$$

In this way, we are indeed implementing a two way dynamical flow of informations between $\Delta^*_{\varepsilon(p)}$ and $\Delta^*_{\varepsilon(q)}$. As a matter of fact, (2.36) provides a concrete mean to associate to each pairwise adjacent set of conformal theories a unique chiral current out of (2.36):

$$\mathbb{W}_{\Omega_{A(p)B(q)}}(z(q,p)) = \begin{cases} \mathbf{W}_{\Omega_{A(p)}}(z(q,p)) & \text{in } \Delta^*_{\varepsilon(p)} \cup \rho_1(p,q) \\ \Omega'^{A(p)B(q)}\mathbf{W}_{\Omega_{B(q)}}(z(q,p)) & \text{in } \Delta^*_{\varepsilon(q)} \cup \rho_1(q,p) \end{cases}. \tag{2.37}$$

We emphasize that, although the second component of $\mathbb{W}_{\Omega_{A(p)B(q)}}(z(q,p))$ should be naturally expressed in term of the coordinate $z(p,q)$, we implicitly exploit the relation $z(q,p) = -z(p,q)$ in order to avoid an unnecessary redundancy.

Eq. (2.37) allows to associate an unique copy of the chiral algebra $\mathcal{W}(p,q)$ to each pairwise adjacent pairs of BCFTs. To this end, let us now consider a small integration contour crossing the (p,q) boundary as in figure 3.

Exploiting the continuity condition along the boundary, the following holds:

$$\begin{aligned} \mathbb{W}_n^{(p,q)} &= \frac{1}{2\pi i}\oint_{C(p,q)} dz(p,q)\ z(p,q)^{n+1}\mathbb{W}_{(p,q)}(z(p,q)) \\ &= \frac{1}{2\pi i}\oint_{C(p)} dz(p,q)\, z(p,q)^{n+1}\mathbf{W}_{\Omega_{A(p)}}(z(p,q)) \\ &\qquad + \frac{1}{2\pi i}\oint_{C(q)} dz(p,q)\, z(p,q)^{n+1}\Omega'^{A(p)B(q)}\mathbf{W}_{\Omega_{B(q)}}(z(p,q)). \end{aligned} \tag{2.38}$$

With eqs. (2.37) and (2.38) we have now introduced all the main ingredients we need in order to coherently define a full-fledged boundary conformal field theory on the whole surface M_∂. As a matter of fact, we can associate to each (p,q) pair of BCFTs defined on cylindrical ends, which are adjacent along a ribbon graph edge, a unique Hilbert space of states $\mathcal{H}^{(p,q)}$; this, can be determined through the action of chiral modes (2.38) on a true vacuum state, whose existence is granted per hypothesis. As usual, $\mathcal{H}^{(p,q)}$ gets decomposed into a direct sum of subspaces $\mathcal{H}_{\lambda(p,q)}$ which are carrier of an irreducible representation of

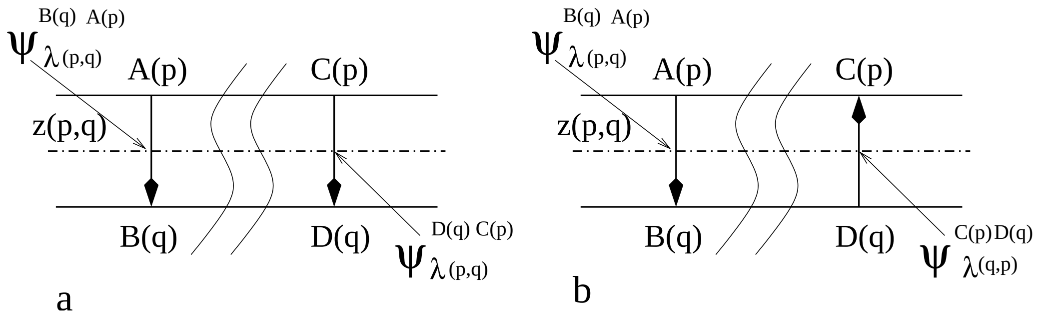

Figure 4: Two-points function of Boundary Insertion Operators.

the $\mathcal{W}(p,q)$ algebra itself. Exploiting the state-to-field correspondence, we can associate to each highest weight state in $\mathcal{H}_{\lambda(p,q)}$ a primary field which we shall refer to as *Boundary Insertion Operator* such that

$$\psi^{A(p)B(q)}_{\lambda(q,p)}(x(q,p)) \,=\, \psi^{B(q)A(p)}_{\lambda(p,q)}(x(p,q)), \tag{2.39}$$

where $x(q,p) = \Re\left[z(q,p)\right]$. In (2.39) the notation is chosen with the following convention: $\lambda(p,q)$ is the representation label, while decoration with indexes $A(p)$ and $B(q)$ points out that the switch in boundary conditions actually refers to all parameters which specify the boundary assignation (to quote, in the case of bosonic string on a toroidal background, it will act both on the glueing automorphism and on the Wilson line/brane position).

Unfortunately, at this stage, BIOs are only purely formal objects. At most, since they are primary operators, we can adopt a description in terms of Chiral Vertex Operators (CVO); in this framework we can interpret them as a map from $\mathcal{H}^{(O)}(p) \otimes \mathcal{H}_{\lambda(p,q)}$, the tensor product between the space of states on $\Delta^*_{\varepsilon(p)}$ with prescribed boundary conditions $A(p)$ and the Fock space of state on the strip into $\mathcal{H}^{(O)}(q)$, the space of states on $\Delta^*_{\varepsilon(q)}$ with prescribed boundary conditions $B(q)$.

In order to "transform" the functions (2.39) in a concretely useful tool from a field theoretical perspective, we must analyze in detail also their analytic and algebraic description. This will be the guiding idea underlying the remaining discussions in this manuscript.

As a starting comment we point out that, since BIOs live on the ribbon graph, their interactions must be guided by the trivalent structure of Γ. Hence, it is useful to summarize here a few results of an exhaustive related analysis on BIOs' correlators [18], which shows how the above introduced geometric structure is sufficient to provide all the fundamental data defining their interaction.

Exploiting the CVO analogue structure, we are able to associate a well-defined conformal dimension to each element as in (2.39) which coincides with the highest weight of the $V_\lambda(p,q)$ module of the Virasoro algebra:

$$H(p,q) \,=\, \frac{1}{2}\lambda^2(p,q). \tag{2.40}$$

Let us deal with the two-point functions between BIOs. To this end, we can exploit explicitly the ribbon graph structure which suggests that they can be introduced as a well-defined concept along any edge $\rho^1(p,q)$ shared between the cylinders $\Delta^*_{\varepsilon(p)}$ and $\Delta^*_{\varepsilon(q)}$. Accordingly, we must take into account two different scenarios: in the first one two operators both mediate a change between boundary conditions in the "p-to-q" direction, while in the second case, one mediates in the "p-to-q", the other in the "q-to-p" (see figure 4).

Out of (2.39) and out of conformal invariance, both kind of correlators have the same analytic form:

$$\begin{aligned}
&\langle \psi^{B(q)A(p)}_{\lambda(p,q)}(x_1(p,q))\psi^{C(p)D(q)}_{\lambda'(q,p)}(x_2(q,p))\rangle = \\
&\langle \psi^{B(q)A(p)}_{\lambda(p,q)}(x_1(p,q))\psi^{C(p)D(q)}_{\lambda'(q,p)}(x_2(q,p))\rangle = \frac{b^{B(q)A(p)}_{\lambda(p,q)}\delta_{\lambda(p,q)\lambda'(p,q)}\delta^{A(p)C(p)}\delta^{B(q)D(q)}}{|x_1(p,q)\,-\,x_2(p,q)|^{2H(p,q)}},
\end{aligned} \tag{2.41}$$

where $H(p,q)$ satisfies the identity (2.40) and where each $b^{B(q)A(p)}_{\lambda(p,q)}$ is a real normalization factor.

Switching now to the three-points function, its structure is mainly driven by the operator product expansion calculated in the hypotheses that BIOs are inserted near any but fixed of the N_2 trivalent vertexes of the ribbon graph (see figure 5). Thus, let us take three points in an infinitesimal open neighborhood with radius ϵ of a vertex $\rho^1(p,q,r)$, chosen as the origin of a suitable local chart [10]. Furthermore let us denote their coordinates as ω_r, ω_p and ω_q and let us focus our attention on three fields $\psi^{A(p)C(r)}_{\lambda(r,p)}(\omega_r)$, $\psi^{B(q)A(p)}_{\lambda(p,q)}(\omega_p)$ and $\psi^{C(r)B(q)}_{\lambda(q,r)}(\omega_q)$ which mediate pairwise the boundary conditions respectively between $\partial\rho^2(r)$ and $\partial\rho^2(p)$, $\partial\rho^2(p)$ and $\partial\rho^2(q)$, $\partial\rho^2(q)$ and $\partial\rho^2(r)$ (as usual, the direction of the action of BIOs is implicitly encoded in the notation).
In the limit $\epsilon \to 0$ the product of the two fields $\psi^{A(p)C(r)}_{\lambda(r,p)}(\omega_r)$ and $\psi^{C(r)B(q)}_{\lambda(q,r)}(\omega_q)$ will mediate the change in boundary conditions from $B(q)$ to $A(p)$. Thus the OPE of these two fields must be expressed as a function of a $\psi^{A(p)B(q)}_{\lambda(q,p)}$-type field:

$$\begin{aligned}
&\psi^{A(p)C(r)}_{\lambda(r,p)}(\omega_r)\psi^{C(r)B(q)}_{\lambda'(q,r)}(\omega_q) \sim \\
&\qquad \sum_{\lambda''(q,p)\in\mathcal{Y}} \mathcal{C}^{A(p)C(r)B(q)}_{\lambda(r,p)\lambda'(q,r)\lambda''(q,p)}|\omega_r\,-\,\omega_q|^{H(q,p)\,-\,H(r,p)\,-\,H(q,r)}\psi^{A(p)B(q)}_{\lambda''(q,p)}(\omega_q),
\end{aligned} \tag{2.42}$$

being $\mathcal{C}^{A(p)C(r)B(q)}_{\lambda(r,p)\lambda'(q,r)\lambda''(q,p)}$ the OPE coefficients. This overall scenario is depicted by the continuous arrows in figure 5. The same holds in all other cases.

Hence the complete description of the interaction among the N_0 different BCFTs requires the determination of an explicit expression for $\mathcal{C}^{A(\cdot)B(\cdot)C(\cdot)}_{\lambda(\cdot,\cdot)\lambda'(\cdot,\cdot)\lambda''(\cdot,\cdot)}$ with A, B, $C \in \mathcal{A}$ and λ, λ', $\lambda'' \in \mathcal{Y}$. Luckily enough this task is partly tractable. As a matter of fact, one can show that both the OPE coefficients and $b^{B(q)A(p)}_{\lambda(p,q)}$ satisfy a set of cyclic properties and sewing constraints that, thanks to the trivalent structure and the variable connectivity of Γ, have an high resemblance with similar problems in ordinary BCFT (see [22]). In particular, in section 4 we will show that, through a suitable choice of toroidal background, it is possible to exploit the variable connectivity of Γ and BIO four-point functions to fix the algebraic form of the above data.

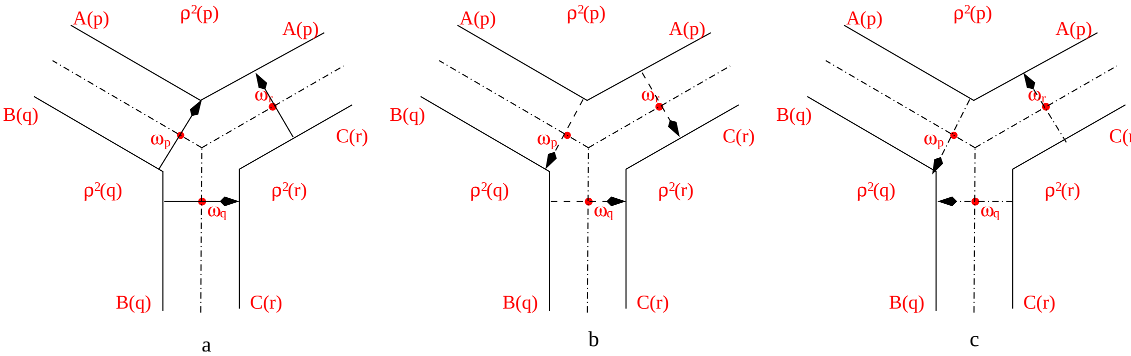

Figure 5: Operator Product Expansion between Boundary Insertion Operators.

3. Open string gauge theory on a RRT

An enhancement of our model, which could also play a pivotal role in gauge/gravity correspondences, calls for the inclusion of open string gauge degrees of freedom (propagating) along the boundaries of M_∂.

To this avail, let us follow usual techniques in open string theory where a non Abelian gauge theory can be naturally included into an open string model by means of a suitable assignation of non-Abelian Chan Paton factors at the open string endpoints.

Thus, let us decorate each $\Delta^*_{\varepsilon(p)}$ with suitable $\mathrm{U}(N)$ Chan-Paton factors (let us remember that M_∂ is oriented). The full string states now transform under the $N\otimes\overline{N}$ representation, namely the adjoint of $\mathrm{U}(N)$. Consequently the generators T^a, $a=1,\ldots,\dim\left[\mathfrak{u}(N)\right]=N^2$ label the string states now belonging to the tensor product between the Fock space associated to the BCFT on the cylinder and the carrier space of the $N\times\overline{N}$ representation i.e. a direct sum of the subspaces[4] $\mathcal{H}_\lambda\otimes N\otimes\overline{N}$ constructed out of the ground state $|0,\lambda;i\bar{j}\rangle$. Conversely we will refer as $(T^a)^i_{\bar{j}}$ to the matrix elements which specify the charges q^i and $q_{\bar{j}}$ created at strings/cylinder endpoints.

In the k-th sector - $k=1,\ldots,N_0$ - the net effect of a dynamical background gauge field A_α is accounted for including in the Polyakov path integral, for each boundary component, a Wilson line term $Tr\left[P\,exp(-S_A)\right]$, where S_A represents the following boundary condensate:

$$S_A=\int d\tau\,A_\alpha\,\partial_\tau X^\alpha. \tag{3.1}$$

Exploiting conformal invariance, the associated β-functions vanish and, in particular, the equation $\beta_A=0$ reduces, at the leading order in the σ-model expansion, to the Yang-Mills equation [23].

The inclusion of gauge degrees of freedom forces us to slightly modify the overall picture on the interaction along the ribbon graph for the BCFTs defined on adjacent cylinders. Since the latter, glued along one edge of the ribbon graph, have opposite orientation, the associated kinematical degrees of freedom must fall into opposite representations of the

[4]The reader should bear in mind that $\mathcal{H}_\lambda$ is still the sub-Hilbert space first appeared in (2.24).

gauge group and, hence, the whole graph acquires a well defined gauge coloring mirroring that of M_∂'s boundary.

Concerning the Fock space for a conformal theory on a shared edge, we can proceed as in the previous section. However we must take into account that, due to the components in the $N \otimes \bar{N}$ space, the states rotate with the action of the adjoint representation of the gauge group, a fundamental datum to take into account whenever we deal with the limit where such states are interpreted as particles. Hence we seek for an object out of the tensor product between the two original Chan-Paton factors thought as elements in the gauge algebra. The only products of this kind are the symmetric and antisymmetric ones between the generators:[5]

$$T^a_{il}(p,q) \;=\; \sum_{b,c=1}^{N^2} \frac{i}{2} f^{abc} \left[T^b_{ij}(p)\,,\, T^c_{jl}(q)\right] \;+\; \sum_{b,c=1}^{N^2} \frac{d^{abc}}{2} \left\{T^b_{ij}(p)\,,\, T^c_{jl}(q)\right\}.$$

Accordingly each BIO belonging to the $\rho^1(p,q)$ BCFT spectrum must be decorated by a $\mathfrak{u}(N)$ generator T^a_{il}. Hence, denoting the conformal structure of BIOs in formula (2.39) with a collective subscript $\Xi(p,q)$, the new non-Abelian BIOs will be matrix-valued functions $\psi^a_{\Xi(p,q)} \doteq T^a \psi_{\Xi(p,q)}$. As a first manifest consequence of these remarks, correlation functions between BIOs acquire a prefactor, namely the trace of the relevant gauge algebra generators.

The first big difference from the analysis in the previous section lies instead in a "simplification" of the two-points function since only the product between two fields mediating along opposite directions is meaningful, hence halving the possible cases.

To summarize, the modified BIOs algebra can be recast (anti)symmetrizing the product of generators:

$$\begin{aligned} \psi^a_{\Xi_1(r,p)}(\omega_r)\,\psi^b_{\Xi_2(q,r)}(\omega_q) \;\sim\; & \frac{1}{2}\sum_{\Xi_3}\sum_{c=1}^{N^2} |\omega_r \,-\, \omega_q|^{H(q,p)\,-\,H(r,p)\,-\,H(q,r)} \times \\ & \left(if^{abc} + d^{abc}\right) \mathcal{C}(\Xi_1(r,p),\Xi_2(q,r),\Xi_3(q,p))\psi^c_{\Xi_3(q,p)}(\omega_q), \end{aligned} \qquad (3.2)$$

being $\mathcal{C}(\Xi_1(r,p),\Xi_2(q,r),\Xi_3(q,p))$ the previously introduced operator algebra fusion coefficients here written with the novel multi-index notation.

3.1 Coupling with background gauge potential

Our next aim is to analyze the new kinematical background emerging after the inclusion of gauge degrees of freedom. Within this respect we will show that the natural coupling with background gauge fields can be rephrased as a move between different orbits in the moduli space of toroidal compactifications. This will allow us to provide a complete characterization of BIOs dynamic specifying the coefficients $\mathcal{C}(\Xi_1(r,p),\Xi_2(p,q),\Xi_3(q,r))$ in (3.2).

[5] Unless stated otherwise, we adopt the following conventions for the algebra structure constants:

$$f^{abc} = \frac{2}{i} Tr\left(\left[T^a, T^b\right] T^c\right), \quad d^{abc} = 2Tr\left(\left\{T^a, T^b\right\} T^c\right).$$

Thus, let us consider a D-dimensional background in which each direction X^α, $\alpha = 0, \ldots, D-1$ is compactified on a circle of radius $\frac{R^\alpha(p)}{l(p)}$. Consistently with the gluing process, let us assume the injection maps obey to arbitrary boundary conditions on the inner boundary of $\Delta^*_{\varepsilon(p)}$ (in the annuli picture). On the outer one, let us assume to have $n+1$ directions satisfying Neumann boundary conditions and $D-n-1$ directions obeying Dirichlet ones:

$$\{X^\alpha\} \doteq \{X^i, X^m\} \quad \text{with } i = 0, \ldots, n, \; m = n+1, \ldots, D-1.$$

To rephrase in a stringy language, we are dealing with a Dn-brane lying along the $X^0, \ldots, X^n$ directions assumed to be coincident with the world-volume parameters $\xi^0, \ldots, \xi^n$ i.e. $\xi^i = X^i$ for all $i = 0, \ldots, n$.

To endow the brane with an interesting dynamic, we have to couple the model to a background gauge field living on its worldvolume: this can be worked out introducing the following boundary action [24, 25]:

$$S_A = \int_0^T d\tau \left[\sum_{i=0}^{n} A_i(\vec{X}) \partial_\tau X^i + \sum_{m=n+1}^{D-1} \phi_m(\vec{X}) \partial_\sigma X^m \right], \tag{3.3}$$

where T is an unspecified (and, at this stage, irrelevant) finite real number, where we have chosen the boundary to lay at constant σ and where $\vec{X} = \{X^0, \ldots, X^n\}$. The $A_i(\vec{X}) = \sum_{a=1}^{N^2} A_i^a(\vec{X}) T^a$ are Lie algebra valued gauge fields on the D-brane, while the entries of the $N \times N$ matrices $\phi_m(\vec{X}) = \sum_{a=1}^{N^2} \phi_m^a(X^i) T^a$ are real scalars from the world volume point of view; the latter describes the motion of the brane in the transverse space.

For the sake of simplicity, let us now assume the brane static in the transverse space i.e. $\phi_m = \mathbf{0}_{N\times N}$ $\forall m = n+1 \ldots D-1$. Moreover let us take constant electric and magnetic fields along the brane worldvolume. Accordingly, the boundary term reads:

$$S_A = \sum_{i,j=0}^{n} F_{ij} \int_0^T d\tau X^j \partial_\tau X^i, \tag{3.4}$$

being F_{ij} the constant field strength out of $A_i(\vec{X})$.

For further convenience, let us specialize to the Abelian subsector. Such specific case can be achieved including in each Neumann direction of the T-dual theory a Wilson line such that $A_i(\vec{X})$ is at the same time a pure gauge and a diagonal matrix i.e. $\mathrm{U}(N)$ symmetry is broken into $\mathrm{U}(1)^N$. At spacetime level, the global effect will be a displacement of the position of N D-branes which, accordingly, entails us to deal only with N separated D-branes.

At a Lagrangian level, on each (p)-subsector the above reasoning translates in the coupling between the open string with a different electromagnetic potential $A_i(p; \vec{X})$;

hence (3.4) is equivalent to:

$$S_{A(p)} \;=\; \sum_{i,j=0}^{n} F_{ij}(p) \int\limits_{\Delta^*_{\epsilon(p)}} d\zeta(p) d\bar{\zeta}(p) \partial X^i(p)\, \bar{\partial}\, \overline{X}^j(p). \tag{3.5}$$

Comparing last formula with (2.18), we can state that, in the Abelian subsector, the net effect of (3.5) is to move to a different point in the flat toroidal background moduli space [26, 27]:

Config. $\mathcal{A}$		Config. $\mathcal{B}$
$G_{\alpha\beta} \;=\; \mathbb{I}_{D\times D}$		$G_{\alpha\beta} \;=\; \mathbb{I}_{D\times D}$
$B_{\alpha\beta} \;=\; 0\, \forall\alpha,\, \beta$	$\Longleftrightarrow$	$B_{\alpha\beta} \;=\; 4\pi\Lambda_{\alpha\beta}$
$F_{\alpha\beta} \;=\; \Lambda_{\alpha\beta}$		$F_{\alpha\beta} \;=\; 0\, \forall\alpha,\, \beta$

On this wise, the description of the new kinematical background directly resides in the choice of a particular point in the moduli space of inequivalent toroidal compactifications in a D-dimensional space, with associated suitable values of the background matrix E entries (see formula (2.19)), namely [23, 28]

$$\mathcal{M} \;=\; O(D,D,\mathbb{Z})\backslash O(D,D)/[O(D)\times O(D)]. \tag{3.6}$$

The different orbits in $\mathcal{M}$ give rise to different theories in which the fundamental $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ current symmetry can be enhanced to different symmetry groups of rank at least D playing the role of gauge group in the target space.

Thus, higher dimensional toroidal compactifications are described by non-trivial background fields B and G and, in such a given background, the maximally enhanced symmetry points are those fixed under the action of $O(D,D,\mathbb{Z})$[6]. In these special points in which (3.7) provides $E' = E$, we can represent extended target space symmetries with respect to a semisimple simply laced Lie algebra of total rank D.

In particular, the maximally enhanced symmetry background can be chosen in the following way [28]: if $C_{\alpha\beta}$, $\alpha,\beta = 1,\ldots,D$, stands for the Cartan matrix of the semisimple simply laced Lie algebra of total rank D, then we must fix:

$$G_{\alpha\beta} \;=\; \frac{1}{2} C_{\alpha\beta} \tag{3.8a}$$

$$B_{\alpha\beta} = G_{\alpha\beta}\; \forall\, \alpha \,>\, \beta, \quad B_{\alpha\beta} \,=\, -\, G_{\alpha\beta} \quad \forall\, \alpha \,<\, \beta, \quad B_{\alpha\alpha} = 0, \tag{3.8b}$$

hence the background matrix $E = G + B \in \mathrm{SL}(D,\, \mathbb{Z})$ and it is fixed under the action (3.7) of $O(D,D,\mathbb{Z})$.

Still in this framework though in a more specific example, let us consider an $O(D,D,\mathbb{Z})$ transformation acting by $M = E^{-1}$ and $\Theta = E^{\dagger} + E$. Hence $E' \;=\; E^{-1}$ and, whenever $G = \mathbb{I}_{D\times D}$ and $B = \mathbf{0}_{D\times D}$, this is exactly the case $(\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)^D$.

[6] $O(D,D,\mathbb{Z})$ is the generalized T-duality group and its action on a background matrix E can be represented in terms of an element $M \in \mathrm{SL}(D,\mathbb{Z})$ and of an antisymmetric integer valued matrix Θ:

$$E' \;=\; M^t\,(E+\Theta)\,M. \tag{3.7}$$

Accordingly, the extended symmetry group associated with the boundary action (3.5) will be:

$$\mathbb{G}_D = (G_{p+1} \times G_{p+1}) \times (\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)^{D-p-1}. \quad (3.9)$$

4. Redefinition of the BCFT in the rational limit

The lesson we can draw from the previous section is the existence of an equivalence between the description of the interaction for each open string with a gauge field living on the brane world-volume and the choice of a point in the moduli space of toroidal compactifications characterized by a non vanishing value of the Kalb-Ramond field.
Moreover we have shown that, among all such choices, we can pick up in $\mathcal{M}_d$ determined as in (3.6) some special points fixed under the action of the generalized T-duality group $O(D, D, \mathbb{Z})$. In these, the emergence of the extended symmetry group (3.9) is an hint of the equivalence at a quantum level between such a theory of compactified D-free scalar bosons and the Wess-Zumino-Witten model associated to the level $k = 1$ simply laced affine algebra $\hat{\mathfrak{g}}$ i.e. the affine extension of the algebra $\mathfrak{g}$ characterized by the Cartan matrix entries (3.8).

In more detail, whenever $E' = E$ out of (3.7), the center of mass string momentum is such that the set of chiral currents of the associated bulk conformal field theory gets enlarged. The new set of arising chiral fields together with the old ones provides for the closure of the $\hat{\mathfrak{g}}$, affine algebra. Moreover, since the Virasoro algebra belongs to the double covering of the chiral one, it is possible to reorganize the infinite sets of highest weights representations into finitely many chiral algebra ones, in particular those appearing when the level k is fixed to 1.

Thus, within this specific choice for the background matrix, in order to analyze what it is the overall behavior of our model with $n+1$ Neumann and $D-n-1$ Dirichlet directions, let us choose a $G_r \times G_r$ factor in (3.9) describing the enhanced symmetry group of a generic but fixed set of r compact directions. According to the previous remark G_r is nothing but the universal covering group generated by exponentiation from the rank-r Lie algebra $\mathfrak{g}$, and this model is equivalent, at a quantum level, to the $\hat{\mathfrak{g}}_1$-WZW model.

We will show that this choice allows us to introduce a specific parametrization of boundary conditions which, through a careful analysis, completely characterize the action of the glueing automorphism introduced in (2.36), hence also the action of Boundary Insertion Operators.

Before getting into the detail of the analysis we shall address in this section, let us fix a few notations and conventions. We denote with $P^k_+(\hat{\mathfrak{g}})$ the set of all finitely many integrable level-k highest weight representations of $\hat{\mathfrak{g}}_k$. The associated highest weights can be characterized by their Dynkin labels (non-negative integers) $\hat{\lambda} \doteq [\lambda_0; \lambda_1, \ldots, \lambda_r] = [\lambda_0, \lambda]$, i.e.[7] their expansion coefficients in the basis of the fundamental weights $\hat{\omega}_l$, $l = 0, \ldots, r$.

[7] In this notation λ denotes the finite part of the weight i.e. it is an integrable highest weight of the parent finite algebra $\mathfrak{g}$. As a side effect let us pinpoint that it does not keep track of the $-L_0$ operator eigenvalue.

$\hat{\mathfrak{g}}$	$\hat{\omega}_I \in P_+^1(\hat{\mathfrak{g}})$	$B(G) \sim \mathcal{O}(\hat{\mathfrak{g}})$	$h^\vee$
$\hat{A}_r$	$\hat{\omega}_0,\, \hat{\omega}_1,\, \ldots,\, \hat{\omega}_r$	$\mathbb{Z}_{r+1}$	$r+1$
$\hat{D}_{r=2l}$	$\hat{\omega}_0,\, \hat{\omega}_1,\, \hat{\omega}_{r-1},\, \hat{\omega}_r$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2r-2$
$\hat{D}_{r=2l+1}$	$\hat{\omega}_0,\, \hat{\omega}_1,\, \hat{\omega}_{r-1},\, \hat{\omega}_r$	$\mathbb{Z}_4$	$2r-2$
$\hat{E}_6$	$\hat{\omega}_0,\, \hat{\omega}_1,\, \hat{\omega}_5$	$\mathbb{Z}_3$	12
$\hat{E}_7$	$\hat{\omega}_0,\, \hat{\omega}_6$	$\mathbb{Z}_1$	18
$\hat{E}_8$	$\hat{\omega}_0$	$\mathbb{I}$	30

Table 1: This table reports fundamental weights belonging to $P_+^1(\hat{\mathfrak{g}})$, the outer automorphism group $\mathcal{O}(\hat{\mathfrak{g}})$, and the dual Coxeter number for $\hat{\mathfrak{g}}$ being a simple laced affine untwisted Lie algebra.

Representations of $\hat{\mathfrak{g}}_k \in P_+^k(\hat{\mathfrak{g}})$ are those satisfying the constraint $k \geq (\hat{\lambda}, \theta)$, where θ is the highest root of $\mathfrak{g}$, while $(\cdot, \cdot)$ is the scalar product naturally induced by its Killing form.

Furthermore we refer to $\chi_{\hat{\lambda}}$ — $\hat{\lambda} \in P_+^k(\hat{\mathfrak{g}})$ — as the characters which carry a representation of the modular group whose properties are partially encoded in $\mathcal{S}^{\text{ext}}$:

$$\chi_{\hat{\lambda}}\left(-\frac{1}{\tau}\right) = \sum_{\hat{\mu} \in P_+^k} \mathcal{S}^{\text{ext}}_{\hat{\lambda}\hat{\mu}}\, \chi_{\hat{\mu}}(\tau).$$

Let us now focus on a specific scenario we are interested in, namely $k = 1$. The only highest weight representations entering in $P_+^1(\hat{\mathfrak{g}})$ are those generated by the highest weights $\hat{\omega}_I$ whose correspondent simple root $\hat{\alpha}_I$ has unit comark. Since the one generated by the basic fundamental weight $\hat{\omega}_0$ always belongs to $P_+^1(\hat{\mathfrak{g}})$, let us rewrite the set of its elements as

$$P_+^1(\hat{\mathfrak{g}}) = \{\hat{\omega}_I\} = \{\hat{\omega}_0, \hat{\omega}_i\}.$$

The explicit set of $\hat{\omega}_I \in P_+^1(\hat{\mathfrak{g}})$ for $\hat{\mathfrak{g}}$ being a simply laced algebra is reported for later convenience and for sake of completeness in table 1.

Within this framework, the bulk theory can then be fully characterized by all the properties of $\hat{\mathfrak{g}}_1$-WZW model. The infinite series of holomorphic and antiholomorphic Verma modules can be reorganized to write the Fock space of the parent bulk theory as the direct sum of the finitely many moduli of the affine Lie algebra:

$$\mathcal{H}^{(C)} = \bigoplus_{\hat{\omega}_I \in P_+^1(\hat{\mathfrak{g}})} \mathcal{H}^{\hat{\mathfrak{g}}_1}_{\hat{\omega}_I} \otimes \overline{\mathcal{H}}^{\hat{\mathfrak{g}}_1}_{\hat{\omega}_I}, \tag{4.1}$$

being $\mathcal{H}^{\hat{\mathfrak{g}}_1}_{\hat{\omega}_I}$ the subHilbert space associated with $\hat{\omega}_I$.

We will denote (the holomorphic part of) the primary fields associated to the highest weight state in $\mathcal{H}^{\hat{\mathfrak{g}}_1}_{\hat{\omega}_I}$ with:[8]

$$\phi_{\hat{I}(p)}(\zeta(p)).$$

[8]From this stage on, we shall trade the subscript $\hat{\omega}_I$ in the operators with $\hat{I}$ in order, hopefully, to provide a simpler notation.

Their components $\phi_{[\hat{I}(p),\,m]}(\zeta(p))$, $m = 1, \ldots, \dim\hat{\omega}_I$, fill the level-0 (in sense of L_0 eigenvalue) subspace of $\mathcal{H}^{\hat{\mathfrak{g}}_1}_{\hat{\omega}_I}$, which we will denote with $\mathcal{V}^0_{\hat{\omega}_I}$. These last subspaces carry an irreducible representation of the horizontal subalgebra of $\hat{\mathfrak{g}}_1$:

$$\mathbf{X}^{\hat{I}}_{J_0} \,:\, \mathcal{V}^0_{\hat{\omega}_I} \rightarrow \mathcal{V}^0_{\hat{\omega}_I}, \tag{4.2}$$

being J_0 a generic element of $\mathfrak{g}$ [20].

Since we are ultimately dealing with rational conformal field theories associated to WZW models, we shall exploit their similarity with conformal minimal models. Thus, to extend them on a surface with boundary, we can adopt Cardy's construction: a set of boundary conditions that we can consistently define on the boundaries are labelled exactly by the modules of the chiral algebra entering into the Hilbert space. The correspondent boundary states are [19]:

$$||\hat{\omega}_I(p)\rangle\rangle \;=\; \sum_{\hat{\omega}_J\,\in\,P^1_+(\hat{\mathfrak{g}})} \frac{\mathcal{S}_{\hat{I}\,\hat{J}}}{\sqrt{\mathcal{S}_{\hat{0}\,\hat{J}}}} |\hat{\omega}_J(p)\rangle\rangle. \tag{4.3}$$

They obey the glueing condition:

$$\left(J^a_n \,+\, \overline{J}^a_{-n}\right) ||\hat{\omega}_I(p)\rangle\rangle \;=\; 0 \qquad\quad \forall\, \hat{\omega}_I \,\in\, P^1_+\,(\hat{\mathfrak{g}}) \tag{4.4}$$

However, Cardy boundary states are not sufficient to describe the plethora of boundary assignations we can coherently fix for a WZW model with a prescribed bulk action. A comprehensive description calls into play deformations techniques of a BCFT [20]. As a matter of fact, when we deal with special points in the toroidal compactifications moduli space, the presence of the enhanced affine symmetry coincides with the presence of new massless *open* string states which can be used to deform the boundary conformal field theory on $\Delta^*_{\varepsilon(p)}$ [27]. In particular, if we pick up among these the chiral deformations, i.e. induced by chiral operators, these deformations are truly marginal (for a brief description of key concepts and techniques in BCFT deformations see [18]), hence the deformed model will change from the undeformed one only for a redefinition of boundary conditions. In this way, starting from an unperturbed Lagrangian, we are able to describe the full set of boundary conditions we can adopt [20] by means of an its suitable deformation.

To provide a detailed description, let us represent the closed affine algebra generators in terms of the boson fields via the Frenkel-Kac-Segal construction of the Weyl-Cartan basis of $\hat{\mathfrak{g}}_{k=1}$. In the closed string channel, the left moving and right moving currents $J^a(\zeta)$, $\bar{J}^a(\bar{\zeta})$ are respectively defined out of following components:

$$H^i(\zeta) \,=\, \partial\, X^i(\zeta), \qquad\qquad E^{\alpha}(\zeta) \;=\; c(\alpha)\; :e^{\,i\sum\limits_{i=1}^{r}\alpha_i X^i(\zeta)}:, \tag{4.5a}$$

$$\overline{H}^i(\bar{\zeta}) \,=\, \sum_{j=1}^{r} M^i_j \bar{\partial}\,\overline{X}^j(\bar{\zeta}), \qquad \overline{E}^{\alpha}(\bar{\zeta}) \;=\; -\bar{c}(\alpha)\; :e^{\,i\sum\limits_{i,j=1}^{r}\alpha_i M^i_j \overline{X}^j(\bar{\zeta})}:. \tag{4.5b}$$

Here $H^i(\zeta)$ and $\overline{H}^i(\bar{\zeta})$ denote the elements in the maximal torus of the two copies of the chiral algebra $\hat{\mathfrak{g}}_{k=1}$, while $\{\alpha\}$ is the set of roots (positives plus negatives) of the parent semi-simple simply laced Lie algebra. The functions $c(\alpha)$ and $\bar{c}(\alpha)$ are $\mathbb{Z}_2$-valued cocycles; these are operators acting on the Fock spaces and they depend only upon the momentum part of the free-boson zero modes. Their inclusion leads the product of the above currents to satisfy the correct OPE [29].

In this framework, the vertex operators associated to the new open string scalar states can be written, in the closed string channel, as

$$S^a_\lambda(u(p))e^{i\sum_{i=1}^{r}\lambda_i X^i(\zeta)} \doteq \frac{1}{2}\left[J^a(\zeta)+\overline{J}^a(\bar\zeta)\right]\, e^{\sum_{i=1}^{r}\lambda_i X^i(\zeta)}\Bigg|_{|\zeta|=\frac{2\pi}{2\pi-\varepsilon(p)}}\,, \tag{4.6}$$

where, $u(p)=\Re\left[\frac{2\pi i}{L(p)}\ln[\zeta(p)]\right]$ is the coordinate parametrizing the inner boundary of $\Delta^*_{\varepsilon(p)}$.

To simplify the notation, from now on, any function dependent upon the coordinate $u(p)$ implicitly refers to the restriction of an holomorphic (or antiholomorphic) map to the locus $|\zeta|=\frac{2\pi}{2\pi-\varepsilon(p)}$.

The occurrence of extra massless open string states in equation (4.6) indicates the enlargement of the chiral algebra of the boundary theory. The associated currents $\mathbf{J}^a(\zeta)$ (which correspond to the vertex operators in (4.6) built on the vacuum representation and generically defined as in equation (2.35) out of the holomorphic and antiholomorphic currents $J^a(\zeta)$ and $\overline{J}^a(\bar\zeta)$), can deform the original theory with a suitable boundary term $S_B = \int du(p)\sum_a g_a\mathbf{J}^a(u(p))$. If we write the currents in the Cartan-Weyl basis, it has been shown in [26] that the most general such a term will be:

$$S'_g = \int_0^{L(p)} du(p)\left(\sum_{\hat\alpha} g_{\hat\alpha}\, e^{i\sum_{i=1}^{r}\hat\alpha_i\, X^i(u(p))} + \sum_{i=1}^{r} g_i\partial_u X^i(u(p))\right), \tag{4.7}$$

where $(g_{\hat\alpha}, g_i)$ are coupling constants and where the new vectors $\hat\alpha$ are related to the simple Lie algebra $\mathfrak{g}$ roots by means of the relation:

$$\alpha_i = \sum_{j=1}^{r}(\delta^i_j + M^i_j)\,\hat\alpha^j \quad \text{where} \quad M = \frac{G+B}{G-B}.$$

Since chiral marginal deformations are truly marginal [20], the deformed model will change from the unperturbed one only for a redefinition of boundary conditions i.e. glueing automorphism and boundary states.

The effect of such a perturbation on the boundary state is a rotation with respect to the left-moving zero modes of the currents [27]:

$$\|B\rangle\!\rangle_g = e^{i\sum_{\hat\alpha} g_{\hat\alpha}E_0^{\hat\alpha} + i\sum_i g_i H_0^i}\,\|B\rangle\!\rangle.$$

Thus, according to the previous formula, it is possible to describe the full set of boundary states of our model through a rotation on a fixed one acting as a “generator” which is associated with the free (unperturbed) model

$$\|g\rangle\!\rangle = g\,\|B\rangle\!\rangle_{(free)} \qquad \Longrightarrow \qquad g = e^{\sum_a g_a J_0^a}. \tag{4.8}$$

They satisfy the perturbed glueing condition:

$$\left[J^a_m \,+\, \gamma_g(\bar{J}^a_{-m})\right] \|g\rangle\!\rangle = 0, \qquad g \,=\, e^{\sum\limits_b g_b \bar{J}^b_0}$$

where $\gamma_g(\bar{J}^a_{-m}) = e^{-\sum\limits_b g_b \bar{J}^b_0} \bar{J}^a_{-m} e^{\sum\limits_b g_b \bar{J}^b_0}$ and they cover the full moduli space of boundary states.

To provide a concrete example, let us consider any but fixed Dirichlet direction. The fixed point in toroidal compactification moduli space is the T-dual radius value, $\frac{R(p)}{l(p)} = \sqrt{2}$, and, as explained at the end of the previous section, the one-boson bulk CFT becomes equivalent to $\hat{\mathfrak{su}}(2)_1$-WZW model. In this case, the free theory has associated Neumann boundary condition with null Wilson line parameter $\tilde{t}_-(p) = 0$, i.e. the free-theory boundary state is $\|N(0)\rangle\!\rangle_{s.d.}$. Hence, the Dirichlet boundary state is obtained exploiting the particular choice of the perturbing boundary action whose associated SU(2) element is $g = e^{-i\pi J^1_0}$:

$$\|D(0)\rangle\!\rangle_{s.d.} \,=\, e^{-i\pi J^1_0} \|N(0)\rangle\!\rangle_{s.d.}. \tag{4.9}$$

Going back to the general case of r directions described through the affine algebra $\hat{\mathfrak{g}}_1$, the boundary action in (4.7) perturbs the spectrum of boundary operators of each independent conformal theory defined on a single cylindrical end. In this connection, the rotation of a boundary field ψ_i induced on a boundary operator $\psi_{\hat{I}(p)}$ (since we are not moving onto a definite representation, we can omit the quantum number m) by a boundary term like that in equation (4.7) is [20]:

$$\tilde{\psi}_{\hat{I}(p)}(u(p)) \,=\, \left[e^{\frac{1}{2}J}\psi_{\hat{I}(p)}\right](u(p)) \,\doteq \sum_{n=0}^{\infty} \frac{\lambda^n}{2^n\, n!} \oint_{C_1} \frac{dv_1}{2\pi} \cdots \oint_{C_n} \frac{dv_n}{2\pi}\, \psi_i(u)\, J(v_1)\, \cdots\, J(v_n), \tag{4.10}$$

where each C_l is a small circle surrounding the J-insertion points. Let us now think at the previous expression as suitably inserted into bulk and boundary fields correlators. Hence we can compute explicitly the expression of $\tilde{\psi}_{\hat{I}(p)}$ thanks to the self locality of boundary operators and to the OPE between the truly marginal fields in the chiral algebra and a boundary operator:

$$\mathbf{J}(u')\psi_{\hat{I}}(u) \sim \frac{\mathbf{X}^{\hat{I}}_{J_0}}{u' - u}\psi_{\hat{I}}(u),$$

where $\mathbf{X}^{\hat{I}}_{J_0}$ is the representation (4.2).

An order by order computation in (4.10) provides

$$\tilde{\psi}_{\hat{I}(p)}(u(p)) \,=\, e^{\frac{i}{2}\mathbf{X}^{\hat{I}}_{J_0}} \psi_{\hat{I}(p)}(u(p)), \tag{4.11}$$

i.e. the natural action of the chiral algebra on the vertex algebra fields translates into the natural action of the representation of an associated element of G_r on the components of a given $\hat{\mathfrak{g}}_1$-module primary field.

Let us now consider two adjacent cylindrical ends, $\Delta^*_{\varepsilon(p)}$ and $\Delta^*_{\varepsilon(q)}$ together with the ribbon graph edge $\rho^1(p,q)$ which they share. Furthermore let us also assume that the theory on the (p)-th polytope is deformed by the action of the boundary term

$S_{B(p)} = \int_0^{L(p)} du(p)\mathbf{J}_1(u(p))$, while the theory on the (q)-th polytope is deformed by $S_{B(q)} = \int_0^{L(q)} du'(q)\mathbf{J}_2(u'(q))$. According to (4.8), the associated boundary states are defined as $\|g_1\rangle\rangle = g_1 \|B\rangle\rangle_{\text{free}}$ and $\|g_2\rangle\rangle = g_2 \|B\rangle\rangle_{\text{free}}$. In such a framework, according to computations in section 2.2, in order to characterize Boundary Insertion Operators it is sufficient to specify the (p,q) glueing automorphism entering in (2.37).

To this avail, the parametrization (4.8) above introduced is not so efficient since it does not allow to successfully explain how the transition between pairwise adjacent boundary conditions takes place.

Thus we need to provide on $\partial\Delta^*_{\varepsilon(p)}$ a new representation for the infinite set of boundary conditions merging the choice of an element within this set with the requirement to have a BIO acting "*à la Cardy*".

As a first step in this direction we prove that Cardy boundary states are those associated to deformations of the unperturbed theory induced by elements in the center $B(G_r)$ of the universal covering group G_r generated by exponentiation of the parent finite algebra $\mathfrak{g}$.

It can be checked case by case that the center $B(G_r)$ is isomorphic to the group of outer automorphisms of the affine algebra $\hat{\mathfrak{g}}$, namely $\mathcal{O}(\hat{\mathfrak{g}})$. It is defined as a regular subgroup of the permutation group $D(\hat{\mathfrak{g}})$ which is nothing but the symmetry group of $\hat{\mathfrak{g}}$ Dynkin diagram [29]. Being $\hat{\mathfrak{g}}$ a simply laced affine algebra, the whole set can be explicitly classified as summarized in table 1. Let us notice that the order of $\mathcal{O}(\hat{\mathfrak{g}})$, and consequently of $B(G_r)$ coincides with the number of moduli of $\hat{\mathfrak{g}}_k$ with $k = 1$. Furthermore the isomorphism between $\mathcal{O}(\hat{\mathfrak{g}})$ and $B(G_r)$ is realized associating to every element $A \in \mathcal{O}(\hat{\mathfrak{g}})$ the element $b \in B(G_r) \hookrightarrow G_r$:

$$b_A = e^{-2\pi i A\hat{\omega}_0\cdot H}, \tag{4.12}$$

where $A\omega_0 \in P^k_+(\hat{\mathfrak{g}})$ and $\hat{\lambda}\cdot H = \sum_i \lambda_i H_0^i - \lambda_0 L_0$. Let us now choose $\|B\rangle\rangle_{\text{free}} \equiv \|\hat{\omega}_0\rangle\rangle$ as a boundary state associated to the free theory.

The perturbation induced by a boundary term associated to $b \in B(G_r)$ acts trivially on the glueing condition because b, defined as in equation (4.12), commutes with all the affine algebra generators J^a_n. Thus $\gamma_b(\overline{J}^a_{-n}) = \overline{J}^a_{-n}$ and the correspondent rotated boundary state must satisfy the unperturbed glueing condition, i.e. it is a Cardy's boundary state.

Moreover, for each element $\hat{\omega}_i \in P^1_+$, it exists a unique element $A_i \in \mathcal{O}(\hat{\mathfrak{g}})$ such that

$$\hat{\omega}_i = A_i\hat{\omega}_0. \tag{4.13}$$

Thus, to complete the proof of our statement, it is sufficient to show that, it holds:

$$\|\hat{\omega}_i\rangle\rangle = b_{A_i}\|\hat{\omega}_0\rangle\rangle \quad \text{with} \quad b_{A_i} = e^{-2\pi i A_i\hat{\omega}_0\cdot H} \in B(G_r). \tag{4.14}$$

Hence let us introduce the following notation for Cardy's boundary states:

$$\|\hat{\omega}_I\rangle\rangle = \sum_J \hat{\omega}_{IJ}|\hat{\omega}_J\rangle\rangle, \tag{4.15}$$

where $\hat{\omega}_{IJ} \doteq \frac{\mathcal{S}^{\text{ext}}_{\hat{I}\hat{J}}}{\sqrt{\mathcal{S}^{\text{ext}}_{\hat{0}\hat{J}}}}$ and where $|\hat{\omega}_J\rangle\rangle$ is the Ishibashi state built upon the $\hat{\omega}_J$-th module.

All descendants in the module $\mathcal{H}^{\hat{\mathfrak{g}}}_{\hat{\omega}_K}$ have the same eigenvalue with respect to b_{A_i} because the generators of the algebra are unaffected by the action of the center:

$$b_{A_i}|\omega'\rangle \;=\; e^{-2\pi i(A_i\hat{\omega}_0,\hat{\omega}'_K)}|\omega'\rangle \;=\; e^{-2\pi i(A_i\hat{\omega}_0,\hat{\omega}_K)}|\omega'\rangle. \quad \forall\,|\omega'\rangle \,\in\, \mathcal{H}^{\hat{\mathfrak{g}}}_{\hat{\omega}_I}$$

The same holds also for Ishibashi states, which are linear combinations of descendant states:

$$b_{A_i}|\hat{\omega}_J\rangle\!\rangle \;=\; e^{-2\pi i(\hat{\omega}_i,\hat{\omega}_J)}|\hat{\omega}_J\rangle\!\rangle \;=\; \begin{cases} |\hat{\omega}_0\rangle\!\rangle & \text{if } J \,=\, 0 \\ e^{-2\pi i F_{ij}}|\hat{\omega}_j\rangle\!\rangle & \text{if } J \,=\, j \end{cases}, \tag{4.16}$$

where $F_{ij} \;=\; (\omega_i,\omega_j)$ is the quadratic form matrix of the parent finite algebra. Since $(\omega_i,\omega_j) = (\hat{\omega}_i,\hat{\omega}_j)$ the proof of (4.14) reduces to verify the following identity:

$$e^{-2\pi i(A_i\hat{\omega}_0,\hat{\omega}_J)}\,\mathcal{S}^{\text{ext}}_{\hat{0}\hat{J}} \;=\; \mathcal{S}^{\text{ext}}_{\hat{\imath}\hat{J}}. \tag{4.17}$$

This last identity holds since the left hand side is the natural action of the automorphism $A_i \in \mathcal{O}(\hat{\mathfrak{g}})$ on the extended modular matrix:

$$e^{-2\pi i(A_i\hat{\omega}_0,\hat{\omega}_J)}\,\mathcal{S}^{\text{ext}}_{\hat{0}\hat{J}} \;=\; \mathcal{S}^{\text{ext}}_{A_i(\hat{0})\hat{J}} \;=\; \mathcal{S}^{\text{ext}}_{\hat{\imath}\hat{J}}.$$

In view of this result, we can exploit the construction in (4.8) to parametrize the generic boundary condition defined over the (inner or outer) boundary of the k-th cylindrical end, represented by the boundary state $\|g(k)\rangle\!\rangle$, with a pair of elements:

$$(\|\hat{\omega}_I\rangle\!\rangle,\,\Gamma(k))\,, \qquad \text{with} \quad \begin{cases} \hat{\omega}_I & \in\; P^1_+(\hat{\mathfrak{g}}) \\ \Gamma(k) & \in\; \frac{G_r}{B(G_r)} \end{cases} \tag{4.18}$$

being $\|\hat{\omega}_I\rangle\!\rangle$ a Cardy's boundary state and $\Gamma(k) \in \frac{G_r}{B(G_r)}$ such that:

$$\|g(k)\rangle\!\rangle \;=\; \Gamma(k)\,\|\hat{\omega}_J(k)\rangle\!\rangle. \tag{4.19}$$

We can show that this parametrization is not only a formal datum. As a matter of fact coset theory ensures that, $\forall\, g \,\in\, G_r$, we can choose a representative $\Gamma \in \frac{G_r}{B(G_r)}$ and an element $b_I \in B(G_r)$ such that g is uniquely decomposed as

$$g \;=\; \Gamma\,\cdot\,b_I. \tag{4.20}$$

Moreover, being $\frac{G_r}{B(G_r)}$ a Lie group,[9] uniqueness of (4.20) allows us to define a global smooth map:

$$\begin{array}{cccccccc} \sigma\colon & \frac{G_r}{B(G_r)} & \longrightarrow & \frac{G_r}{B(G_r)} \times B(G_r) & \hookrightarrow & G_r \\ & \Gamma & \mapsto & (\Gamma,\,e) & \mapsto & \Gamma\,\cdot\,e \end{array}, \tag{4.21}$$

[9]The converse to Lie third theorem [30] ensures that, given a finite dimensional abstract real Lie algebra $\mathfrak{g}$, there is a single *simply connected* Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$, namely the *universal covering group* generated by $\mathfrak{g}$. All other groups with the same Lie algebra can be obtained from the universal covering one by quotient with one of its invariant discrete subgroups - say D.

The factor group $H = \frac{G}{D}$ is a multiply connected Lie group since, quoting from [31], it holds: **Proposition 1.** *Let G be a Lie group with center $B(G)$ such that $D \subseteq B(G)$ is a finite subgroup of G. Then there is a unique Lie group structure on the quotient group $H = G/D$ such that the quotient map $G \to H$ is a Lie group map.*

where e is the identity of G_r. Hence, since all the hypotheses of proposition 1 are met and since σ is also a group homomorphism, we can consider $\left(\frac{G_r}{B(G_r)}, \sigma\right)$ a Lie subgroup of G_r. The inclusion $\sigma : \frac{G_r}{B(G_r)} \hookrightarrow G_r$ translates at a level of Lie algebras as

$$d\sigma : \quad \mathfrak{g} \to \mathfrak{g}. \tag{4.22}$$

The following holds [32]:

Proposition 2. *Let (H, σ) be a Lie subgroup of G_r with Lie algebra $\mathfrak{h}$ and let $X \in \mathfrak{g}$. If $X \in d\sigma(\mathfrak{h})$, then $e^{tX} \in \sigma(H)$ for all $t \in \mathbb{R}$. Conversely, if $e^{tX} \in \sigma(H)$ for t in some open real interval, then $X \in d\sigma(\mathfrak{h})$.* (Proof can be found at pg. 104 section 3 in [32]).

According to this last proposition, the existence of a (global) exponential map from the image of (4.22) into $\frac{G_r}{B(G_r)}$ is granted, i.e. for any $\Gamma \in \frac{G_r}{B(G_r)}$, we can uniquely write the immersion $\sigma(\Gamma) \in G_r$ as

$$\sigma(\Gamma) \equiv \Gamma \;=\; e^{i \sum_a \Gamma_a J^a},$$

where $\{J^a\} \doteq \{E_0^\alpha, H_0^r\}$ are the (Cartan-Weyl) generators of the horizontal subalgebra of $\hat{\mathfrak{g}}_1$ isomorphic to $\mathfrak{g}$. Let us denote the element $b_I \in B(G_r)$ as $b_I \doteq e^{i b_r H_0^r}$; the Baker-Campbell-Hausdorff formula [30, 31] ensures that it holds a precise relation among coefficients b_r, Γ_a and g_a such that we can write:

$$g \;=\; e^{i \sum_a g_a J_0^a} \;=\; e^{i \sum_a \Gamma_a J^a}\, e^{i b_r H_0^r}. \tag{4.23}$$

The associated boundary state $\|g\rangle\rangle$ can be uniquely written as

$$\|g(k)\rangle\rangle \;=\; g\, \|\hat{\omega}_0\rangle\rangle \;=\; \Gamma \cdot b_I\, \|\hat{\omega}_0\rangle\rangle \;=\; \Gamma\, \|\hat{\omega}_I\rangle\rangle.$$

Hence we have split the deformation process in (4.7)-(4.8) in two subsequent steps. The first involves a deformation induced by a boundary action term $\mathcal{S}_b = \int du(p)\, [\sum_r b_r H^r(u(p))]$, uniquely determined by a group element in the center of the universal covering group, $B(G_r)$. This deformation actually maps the old free boundary state into a Cardy one, while its action changes the boundary operators only for a multiplication of their components by a constant phase factor.

The second step is instead a deformation induced by the boundary term $\mathcal{S}_\Gamma(p) = \int du(p) \sum_a \Gamma_a J^a(u(p))$, which acts on a Cardy boundary state mapping it into $\|g\rangle\rangle = \Gamma\, \|\hat{\omega}_I\rangle\rangle$; at the same time it can act non-trivially on boundary operators.

To summarize, the above parametrization states that we are actually performing a deformation of $\hat{\mathfrak{g}}_1$-WZW model described “à la Cardy” by means of a boundary term such that the associated group element $e^{i \sum_a \Gamma_a J_0^a}$ is the image of $\Gamma \in \frac{G_r}{B(G_r)}$ in G_r by means of the map σ.

Within this framework, the amplitude intermediate channels associated with the generic (p, q)-edge of the ribbon graph, will correspond to an automorphism induced by the operator $g_1(p) g_2^{-1}(q)$. Parametrization (4.18) allows to “make explicit” the action of this glueing automorphism (2.37) with two separate objects: a map which relates dynamically the two Cardy boundary states and an action on the residues $\frac{G_r}{B(G_r)}$ deformations induced by Γ_1 and Γ_2. The former is easily retrieved reasoning in analogy with the definition of boundary conditions changing operators of rational minimal models. As a matter of fact we can define this "first act" of the glueing process as the fusion between the representations associated to the two adjacent Cardy's boundary states and the one a BIO carries. Thus, let us consider the (p,q) edge $\rho^1(p,q)$, and boundary conditions specified uniquely by the central actions $S_b(p)$ and $S_b(q)$. If $\|\hat{\omega}_J(p)\rangle\rangle$ and $\|\hat{\omega}_J(q)\rangle\rangle$ are the associated boundary states, which are shared by $\rho^1(p,q)$, BIOs on $\rho^1(p,q)$ are defined as

$$\psi^{\hat{J}(p)\,\hat{J}(q)}_{\hat{I}(p,q)}(x(p,q)) \;=\; \mathcal{N}^{\hat{J}(q)}_{\hat{J}(p)\,\hat{I}(p,q)}\,\psi_{\hat{I}(p,q)}(x(p,q)), \tag{4.24}$$

i.e. they are the $\hat{g}_{k=1}$ primary fields weighted by the fusion rule $N^{\hat{J}(q)}_{\hat{J}(p)\,\hat{I}(p,q)}$. These are provided by a combination of the $\mathcal{S}^{\mathrm{ext}}$ matrix entries via the Verlinde formula

$$\mathcal{N}^{\hat{J}(q)}_{\hat{J}(p)\,\hat{I}(p,q)} \;=\; \sum_{\hat{\omega}_K\in P^1_+(\hat{\mathfrak{g}})} \frac{S^{\mathrm{ext}}_{\hat{J}(p)\,\hat{K}}\,S^{\mathrm{ext}}_{\hat{I}(p,q)\,\hat{K}}\,\overline{S}^{\mathrm{ext}}_{\hat{K}\,\hat{J}(q)}}{S^{\mathrm{ext}}_{\hat{0}\,\hat{K}}}. \tag{4.25}$$

At this stage the "second act" is straightforward. Let us switch on the boundary terms $S_{\Gamma_1}(p)$ and $S_{\Gamma_2}(q)$ on the inner boundaries of $\Delta^*_{\varepsilon(p)}$ and $\Delta^*_{\varepsilon(q)}$. We can deform the (p,q) theory with a suitable combination of currents which maps the $S_{\Gamma_1}(p)$-induced deformation into the $S_{\Gamma_2}(q)$-induced deformation. In the forthcoming analysis, we will show that this choice allows the two ends to glue dynamically in such a way that such a dynamic is actually governed by the fusion rules of the WZW model. The explicit expression of this combination of currents is established by requiring that the image into G_r of the associated group element in $\frac{G_r}{B(G_r)}$ is $\overline{\Gamma} \;=\; \Gamma_2\Gamma_1^{-1}$. Viceversa, if we consider the (q,p) theory - formally distinct form the (p,q)-one -, the image of the associated element would be $\overline{\Gamma}^{-1} \;=\; \Gamma_1\Gamma_2^{-1}$. Thus let us write the desired defect term as

$$S_{(p,q)} \;=\; \int\limits_{\rho^1(p,q)} dx(p,q)\,\sum_{a}^{\dim\mathfrak{g}}\overline{\Gamma}_a\mathbb{J}_{a\,(p,q)}(x(p,q)), \tag{4.26}$$

where $\mathbb{J}_{a\,(p,q)}$ is defined as in (2.37). The combination of (2.37) and (2.35) allows to write the above defect term exactly as a boundary perturbing term for the $\Delta^*_{\varepsilon(p)}$ theory (let us remember that the formal expression in (2.37) has to be defined separately for the holomorphic and antiholomorphic components): it maps the boundary state in $\Gamma_1\,\|\hat{\omega}_I\rangle\rangle$ into $\Gamma_2\,\|\hat{\omega}_I\rangle\rangle$.

To describe the effect of (4.26) on Boundary Insertion Operators, let us consider the functional expression of their components, dropping the dependence from the fusion rule factor:

$$\psi_{[\hat{I},m](p,q)}, \qquad \text{with} \quad \hat{\omega}_I \,\in\, P^1_+. \quad m \,=\, 1,\,\ldots,\,\dim|\hat{\omega}_I|$$

Since functional and conformal properties of Boundary Insertion Operators are strictly analogue to those of ordinary boundary operators (see section 2.2), we can apply to the formers exactly the same arguments as in equation (4.10) and subsequents. Hence the defect term will deform BIOs by means of a rotation:

$$\psi_{[\hat{I},m](p,q)} \longrightarrow e^{\frac{i}{2}X^{\hat{I}}_{\Gamma}}\, \psi_{[\hat{I},m](p,q)},$$

i.e. the action of the chiral algebra translates into the action of the associated group via its unitary representations. Thus, restoring the fusion coefficients, we have the following expression for boundary insertion operators in the rational limit of the conformal theory:

$$\psi^{[\hat{J}_2,\Gamma_2](q)\,[\hat{J}_1,\Gamma_1](p)}_{[\hat{I},m](p,q)} = \sum_{n=0}^{\dim|\hat{I}|} R^{\hat{I}(p,q)}_{m\,n(p,q)}(\Gamma_2{\Gamma_1}^{-1})\, \psi^{\hat{J}_2(q)\,\hat{J}_1(p)}_{[\hat{I},n](p,q)}, \tag{4.27}$$

where $\psi^{\hat{J}(p)\,\hat{J}(q)}_{\hat{J}(p,q)}(x(p,q)) = \mathcal{N}^{\hat{J}(q)}_{\hat{J}(p)\,\hat{I}(p,q)}\,\psi_{\hat{I}(p,q)}(x(p,q))$ and where $R^{\hat{I}(p,q)}_{m\,n(p,q)} = \exp\left[\frac{i}{2}X^{\hat{I}(p,q)}\right]_{m\,n(p,q)}$ being $X^{\hat{I}}$ the operator introduced in (4.2).

4.1 The algebra of boundary insertion operators

The aim of this rather technical section is to show that, with boundary insertion operators defined as in equation (4.27), boundary perturbations do not affect the algebra of boundary operators which is completely fixed in terms of the fusion rules of the WZW-model. This is indeed a check that our prescription for the (p,q) glueing automorphism and its consequent action on BIOs is consistent: as a matter of fact all deformations we have introduced are actually truly marginal ones and, thus, they must not break the chiral symmetry generated by $\hat{\mathfrak{g}}$.

The algebra of rotated BIOs follows from their definition. Let us notice that rotated BIOs are just a superposition of the different components of Cardy $\hat{\mathfrak{g}}_1$-chiral primary operators' components.

Let us focus our attention on the two-point function between a p-to-q and q-to-p mediating operators.[10] We have to compute:

$$\left\langle \psi^{[\hat{J}_2,\Gamma_2](q)\,[\hat{J}_1,\Gamma_1](p)}_{[\hat{I},m](p,q)}(x_1(p,q))\,\psi^{[\hat{J}_3,\Gamma_3](p)\,[\hat{J}_4,\Gamma_4](q)}_{[\hat{I}',m'](q,p)}(x_2(q,p))\right\rangle . \tag{4.28}$$

As a first step we must notice that a coherent glueing imposes the two operators to mediate between the same boundary conditions (see equation (2.41)). Accordingly the above expression reduces to:

$$\begin{aligned}\left\langle \psi^{[\hat{J}_2,\Gamma_2](q)\,[\hat{J}_1,\Gamma_1](p)}_{[\hat{I},m](p,q)}(x_1(p,q))\,\psi^{[\hat{J}_1,\Gamma_1](p)\,[\hat{J}_2,\Gamma_2](q)}_{[\hat{I}',m'](q,p)}(x_2(q,p))\right\rangle &= \\ \sum_{n\,n'} R^{\hat{I}(p,q)}_{m\,n(p,q)}(\Gamma_2{\Gamma_1}^{-1}) R^{\hat{I}'(q,p)}_{m'\,n'(q,p)}(\Gamma_1{\Gamma_2}^{-1})\times& \\ \left\langle \psi^{\hat{J}_2(q)\,\hat{J}_1(p)}_{[\hat{I},n](p,q)}(x_1(p,q))\,\psi^{\hat{J}_1(p)\,\hat{J}_2(q)}_{[\hat{I}',n'](q,p)}(x_2(q,p))\right\rangle .&\end{aligned} \tag{4.29}$$

[10] Let us remember that the other possible two-points function loses its physical meaning after a suitable assignation of Chan-Paton factors (see comments at the end of section 3).

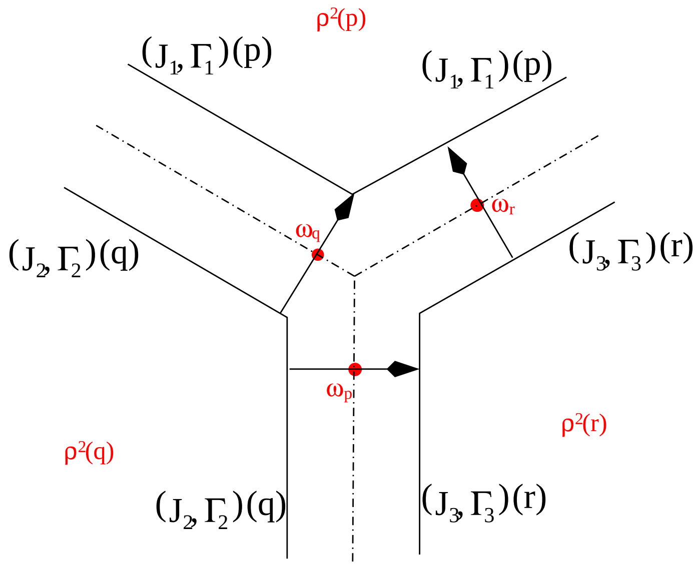

Figure 6: OPE between rotated Boundary Insertion Operators.

Let us notice that, in the previous expression, we are dealing with a representation of the diagonal subgroup of the direct product $\frac{G_r}{B(G_r)}(p,q) \times \frac{G_r}{B(G_r)}(q,p)$; hence it holds (see eq. (A.1)):

$$R^{\hat{I}(p,q)}_{m\,n(p,q)}(\Gamma_2{\Gamma_1}^{-1}) R^{\hat{I}'(q,p)}_{m'\,n'(q,p)}(\Gamma_1{\Gamma_2}^{-1}) \;=\; R^{\hat{I}\times\hat{I}'}_{m\,n;\,m'\,n'}(\mathbb{I}). \tag{4.30}$$

The Clebsh-Gordan expansion (eq. (A.2)) gives (we omit the polytope indexes in the Clebsh-Gordan coefficients):

$$\begin{aligned}&\left\langle \psi^{[\hat{J}_2,\,\Gamma_2](q)\,[\hat{J}_1,\,\Gamma_1](p)}_{[\hat{I},\,m](p,q)}(x_1(p,q))\,\psi^{[\hat{J}_1,\,\Gamma_1](p)\,[\hat{J}_2,\,\Gamma_2](q)}_{[\hat{I}',\,m'](q,p)}(x_2(q,p))\right\rangle \;= \\ &\qquad \sum_{n\,n'}\sum_{\hat{J}\,N} C^{\hat{J}\,N}_{\hat{I}\,m\,\hat{I}'\,m'} C^{\hat{J}\,N}_{\hat{I}\,n\,I'\,n'} \left\langle \psi^{\hat{J}_2(q)\,\hat{J}_1(p)}_{[\hat{I},\,n](p,q)}(x_1(p,q))\,\psi^{\hat{J}_1(p)\,\hat{J}_2(q)}_{[\hat{I}',\,n'](q,p)}(x_2(q,p))\right\rangle \;= \\ &\qquad\qquad \left\langle \psi^{\hat{J}_2(q)\,\hat{J}_1(p)}_{[\hat{I},\,m](p,q)}(x_1(p,q))\,\psi^{\hat{J}_1(p)\,\hat{J}_2(q)}_{[\hat{I}',\,m'](q,p)}(x_2(q,p))\right\rangle,\end{aligned} \tag{4.31}$$

where, in the last equation, we have used the completeness of Clebsh-Gordan coefficients (see equation (A.3b)).

To calculate the OPE of rotated BIOs, let us notice that the rotation generated by the boundary condensate does not change the coordinate dependence. Let us consider the situation depicted in figure 6.

OPE between $\psi^{[\hat{J}_1,\,\Gamma_1](p)\,[\hat{J}_3,\,\Gamma_3](r)}_{[\hat{I}_1,\,m_1](r,p)}$ and $\psi^{[\hat{J}_3,\,\Gamma_3](r)\,[\hat{J}_2,\,\Gamma_2](q)}_{[\hat{I}',\,m'](q,r)}$ will mediate a change in boundary conditions from $[\hat{J}_2,\,\Gamma_2](q)$ to $[\hat{J}_1,\,\Gamma_1](p)$. In particular,

$$\begin{aligned}&\psi^{[\hat{J}_1,\,\Gamma_1](p)\,[\hat{J}_3,\,\Gamma_3](r)}_{[\hat{I}_1,\,m_1](r,p)}(\omega_r)\,\psi^{[\hat{J}_3,\,\Gamma_3](r)\,[\hat{J}_2,\,\Gamma_2](q)}_{[\hat{I}_2,\,m_2](q,r)}(\omega_q) \;= \\ &\qquad \sum_{n_1(r,p)\,n_2(q,r)} R^{\hat{I}_1(r,p)}_{m_1\,n_1(p,q)}(\Gamma_1{\Gamma_3}^{-1})\,R^{\hat{I}_2(q,r)}_{m_2\,n_2(q,r)}(\Gamma_3{\Gamma_2}^{-1})\,\psi^{\hat{J}_1(p)\,\hat{J}_3(r)}_{[\hat{I}_1,\,n_1](r,p)}(\omega_r)\,\psi^{\hat{J}_3(r)\,\hat{J}_2(q)}_{[\hat{I}_2,\,m_2](q,r)}(\omega_q).\end{aligned}$$

We are dealing again with a representation of the diagonal subgroup of the direct product $\frac{G_r}{B(G_r)}(r,p) \times \frac{G_r}{B(G_r)}(q,r)$; hence, applying (A.1) and the Clebsh-Gordan series expansion (A.2), we are left with

$$\sum_{\substack{n_1(r,p)\\ n_2(q,r)}} \sum_{\hat{I}} \sum_{m,n=1}^{\dim \hat{I}} C^{\hat{I}\,m}_{\hat{I}_1(r,p)\,m_1(r,p)\,\hat{I}_2(q,r)\,m_2(q,r)}\, R^{\hat{I}}_{m\,n}(\Gamma_1\Gamma_2^{-1}) \times C^{\hat{I}\,n}_{\hat{I}_1(r,p)\,n_1(r,p)\,\hat{I}_2(q,r)\,n_2(q,r)}\, \psi^{\hat{J}_1(p)\,\hat{J}_3(r)}_{[\hat{I}_1,\,n_1](r,p)}(\omega_r)\, \psi^{\hat{J}_3(r)\,\hat{J}_2(q)}_{[\hat{I}_2,\,n_2](q,r)}(\omega_q). \tag{4.32}$$

According to (2.42), the OPE between undeformed Boundary Insertion Operators reads:

$$\psi^{\hat{J}_1(p)\,\hat{J}_3(r)}_{[\hat{I}_1,\,n_1](r,p)}(\omega_r)\, \psi^{\hat{J}_3(r)\,\hat{J}_2(q)}_{[\hat{I}_2,\,n_2](q,r)}(\omega_q) = \sum_{\hat{I}_3,n_3} |\omega_r - \omega_q|^{H(q,p)-H(r,p)-H(q,r)} C^{\hat{I}_3\,n_3}_{\hat{I}_1\,n_1\,\hat{I}_2\,n_2}\, \mathcal{C}^{\hat{J}_1(p)\,\hat{J}_3(r)\,\hat{J}_2(q)}_{\hat{I}_1\,\hat{I}_2\,\hat{I}_3}\, \psi^{\hat{J}_1(p)\,\hat{J}_2(q)}_{[\hat{I}_3,\,n_3](q,p)}(\omega_q), \tag{4.33}$$

where the Clebsh-Gordan coefficients $C^{\hat{I}_3\,n_3}_{\hat{I}_1\,n_1\,\hat{I}_2\,n_2}$ compensate the fact that the l.h.s. and r.h.s. terms have different transformation behavior under the action of the horizontal $\mathfrak{g}$ algebra, while the coefficients $\mathcal{C}^{\hat{J}_1(p)\,\hat{J}_3(r)\,\hat{J}_2(q)}_{\hat{I}_1\,\hat{I}_2\,\hat{I}_3}$ reflect the non trivial dynamic on each trivalent vertex of the ribbon graph.

The inclusion of this last OPE into (4.32) and the Clebsh-Gordan coefficients unitarity (equation (A.3b)) leaves us with:

$$\psi^{[\hat{J}_1,\,\Gamma_1](p)\,[\hat{J}_3,\,\Gamma_3](r)}_{[\hat{I}_1,\,m_1](r,p)}(\omega_r)\, \psi^{[\hat{J}_3,\,\Gamma_3](r)\,[\hat{J}_2,\,\Gamma_2](q)}_{[\hat{I}_2,\,m_2](q,r)}(\omega_q) = \sum_{j_3\,m} C^{\hat{I}_3\,m}_{\hat{I}_1\,m_1\,\hat{I}_2\,m_2}\, \mathcal{C}^{\hat{J}_1(p)\,\hat{J}_3(r)\,\hat{J}_2(q)}_{\hat{I}_1\,\hat{I}_2\,\hat{I}_3}\, \psi^{[\hat{J}_1,\,\Gamma_1](p)\,[\hat{J}_2,\,\Gamma_2](q)}_{[\hat{I}_3,\,m_3](q,p)}(\omega_p). \tag{4.34}$$

Thus, we demonstrated that OPE between rotated BIOs is formally equal to OPE between unrotated ones. Accordingly, on the ribbon graph the non trivial dynamic is given by the fusion among the three representations entering in each trivalent vertex.

This allows us to further pursue our investigation and to consider the four-points function between BIOs included on graph edges which are among four adjacent polytopes:

$$\langle \psi^{\hat{J}_1(p)\,\hat{J}_4(s)}_{\hat{I}_1(s,p)}\, \psi^{\hat{J}_4(s)\,\hat{J}_3(r)}_{\hat{I}_2(r,s)}\, \psi^{\hat{J}_3(r)\,\hat{J}_2(q)}_{\hat{I}_3(q,r)}\, \psi^{\hat{J}_2(q)\,\hat{J}_1(p)}_{\hat{I}_4(p,q)} \rangle. \tag{4.35}$$

The variable connectivity of the triangulation becomes fundamental in this computation since it allows to state a correspondence between the two possible factorizations out of which we can compute (4.35) and the two ways we can fix adjacency of the four polytopes involved in the analysis.

Let us consider the natural picture in which we construct a four-points function arises, namely the neighborhood of two near trivalent vertexes. Due to the variable connectivity of the triangulation, the two configurations shown in figure 7 are both admissible. The

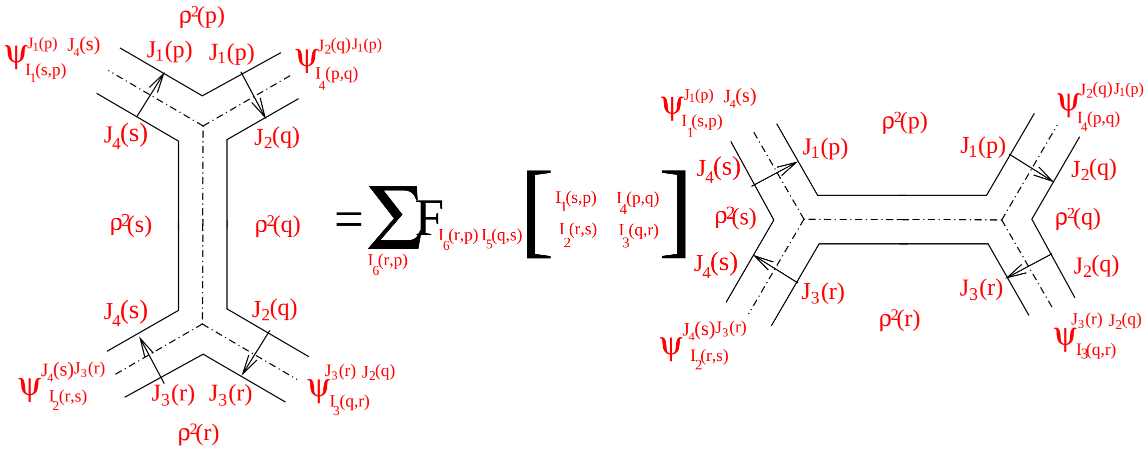

Figure 7: Four-points function crossing symmetry.

transition from the situation depicted in the l.h.s. and the one in the r.h.s. of the pictorial identity in figure 7 corresponds exactly to the transition between the s-channel and the t-channel of the four-points blocks of a single copy of the bulk theory.

The two factorizations of the above four-points function are related by the bulk crossing matrices:

$$F_{\hat{I}_6(s,q)\,\hat{I}_5(r,p)} \begin{bmatrix} \hat{I}_4(p,s) & \hat{I}_1(q,p) \\ \hat{I}_3(s,r) & \hat{I}_2(r,q) \end{bmatrix}. \tag{4.36}$$

The explicit computation of the two factorizations leads to the relation

$$\begin{aligned} &\mathcal{C}^{\hat{J}_4(s)\,\hat{J}_3(r)\,\hat{J}_2(q)}_{\hat{I}_2(r,s)\,\hat{I}_3(q,r)\,\hat{I}_5(q,s)}\,\mathcal{C}^{\hat{J}_1(p)\,\hat{J}_4(s)\,\hat{J}_2(q)}_{\hat{I}_1(s,p)\,\hat{I}_5(q,s)\,\hat{I}_1(s,p)}\,\mathcal{C}^{\hat{J}_1(p)\,\hat{J}_2(q)\,\hat{J}_1(p)}_{\hat{I}_1(s,p)\,\hat{I}_1(s,p)\,0} = \\ &\qquad \sum_{\hat{I}_5(r,p)} F_{\hat{I}_6(s,q)\,\hat{I}_5(r,p)} \begin{bmatrix} \hat{I}_4(p,s) & \hat{I}_1(q,p) \\ \hat{I}_3(s,r) & \hat{I}_2(r,q) \end{bmatrix} \times \\ &\qquad\qquad \mathcal{C}^{\hat{J}_1(p)\,\hat{J}_4(s)\,\hat{J}_3(r)}_{\hat{I}_1(s,p)\,\hat{I}_2(r,s)\,\hat{I}_6(r,p)}\,\mathcal{C}^{\hat{J}_3(r)\,\hat{J}_2(q)\,\hat{J}_1(p)}_{\hat{I}_3(q,r)\,\hat{I}_4(p,q)\,\hat{I}_6(p,r)}\,\mathcal{C}^{\hat{J}_1(p)\,\hat{J}_3(r)\,\hat{J}_1(p)}_{\hat{I}_6(r,p)\,\hat{I}_6(p,r)\,0}, \end{aligned} \tag{4.37}$$

i.e. the usual BCFT sewing relation among boundary operators OPEs.

This statement completes our analysis of the conformal properties of the full theory arising glueing together the BCFTs defined over each cylindrical end; within the above construction BIOs play exactly the same role as the usual boundary operators in BCFT.

This analogy allows us to apply to BIOs all boundary operators properties. In particular, we can identify their OPE coefficients describing interactions in the neighborhood of the (p, q, s) vertex of the ribbon graph with the fusion matrices (4.36) with the following entries assignations:

$$\mathcal{C}^{\hat{J}_1(p)\,\hat{J}_2(q)\,\hat{J}_3(s)}_{\hat{I}_1(q,p)\,\hat{I}_2(s,q)\,\hat{I}_3(s,p)} = F_{\hat{J}_2(q)\,\hat{I}_3(s,p)} \begin{bmatrix} \hat{J}_1(p) & \hat{J}_3(s) \\ \hat{I}_1(q,p) & \hat{I}_2(s,q) \end{bmatrix}. \tag{4.38}$$

Relation (4.38), first obtained in [33] for the A-series minimal models, has been recast for all minimal models and extended rational conformal field theories in [34] and [35] exploiting the full analogy between equation (4.37) and the pentagonal identity for the fusing matrices.

According to [36], WZW-models fusion matrices coincide with the $6j$-symbols of the corresponding quantum group with deformation parameter given by the $(k + g^{\vee})$-th root of the identity, where k and $h^{\vee}$ are respectively the level and the dual Coxeter number of the extended algebra (the list of dual Coxeter numbers for the rank-r simply laced algebras can be found in table 1). Thus, with $k = 1$, the OPEs coefficients are the quantum group $G_{Q=e^{\frac{2\pi i}{1+h^{\vee}}}}$ $6j$-symbols:

$$\mathcal{C}_{\hat{I}_1(q,p)\,\hat{I}_2(s,q)\,\hat{I}_3(s,p)}^{\hat{J}_1(p)\,\hat{J}_2(q)\,\hat{J}_3(s)} = \begin{Bmatrix} \hat{I}_1(q,p) & \hat{J}_1(p) & \hat{J}_2(q) \\ \hat{J}_3(s) & \hat{I}_2(s,q) & \hat{I}_3(s,p) \end{Bmatrix}_{Q=e^{\frac{2\pi i}{1+h^{\vee}}}}. \tag{4.39}$$

5. Discussion and conclusions

In this paper we have fully characterized the local coupling between a scalar Rational Boundary Conformal Field Theory and a special class of open surfaces M_{∂}. The latter arise as uniformizations of a Random Regge Triangulation. In this connection, the results in previous section provide the main ingredient needed to write a worldsheet amplitude defined over the full M_{∂}. To this aim, a possible candidate is a construction introduced in [11]. Exploiting an edge vertex factorization of the most general BIOs' correlator we can write on the ribbon graph, we can express the contribution to the graph-amplitude given by each set of r fields $\{X^i\}, i = 1, \dots, r$ associated to each factor entering in (3.9) as

$$Z(\Gamma, r) = \sum_{\{\hat{I}(r,p)\}\in P_1^+\hat{\mathfrak{g}}} \prod_{\rho^0(p,q,r)=1}^{N_2(T)} \mathcal{C}_{\hat{I}_1(r,p)\hat{I}_2(q,r)\hat{I}_3(q,p)}^{\hat{J}_1(p)\hat{J}_3(r)\hat{J}_2(q)} \times \prod_{\rho^1(p,r)=1}^{N_1(T)} \left(b_{\hat{I}_1(r,p)}^{\hat{J}_1(p)\hat{J}_3(r}\right)\Big)^2 L(p,r)^{-2H_{\hat{I}_1(r,p)}}. \tag{5.1}$$

The sum runs over all the $N_1(T)$ primaries of the chiral algebra decorating the ribbon graph edges through the insertion of BIOs, and with the OPE coefficients $\mathcal{C}_{\hat{I}_1(r,p)\hat{I}_2(q,r)\hat{I}_3(q,p)}^{\hat{J}_1(p)\hat{J}_3(r)\hat{J}_2(q)}$ being replaced by the associated $6j$ symbols.

Afterward each contribution must be applied on the associated $N_0(T)$ channels defined by the cylinder amplitude for the correspondent directions. Concerning any but fixed factor in eq. (3.9), we want to define the transition amplitude between two boundary states $\|g_1\rangle\rangle$ and $\|g_2\rangle\rangle$, the latter being constructed out of the action of an element $g \in G_r$ on the first one: $\|g_2\rangle\rangle = g\,\|g_1\rangle\rangle$. As proved in details in section 4 of [20] or [37], it is easy to show that the amplitude $\mathcal{A}_{\Delta^*_{\varepsilon(p)}}^{g_1;\,g\cdot g_1}$ depends only upon the conjugacy classes of $g \in G_r$. Therefore, we can choose to deform the boundary state with an element in the maximal torus of G_r, $h = e^{i\sum_{i=1}^r \lambda_i H^i}$. Thus, if we choose $\|g_1\rangle\rangle$ to coincide with one of the Cardy boundary states $\|\hat{\omega}_K\rangle\rangle$,[11] the amplitude will involve a sum over $\hat{\mathfrak{g}}_1$ characters, twisted by the action

[11] Let us notice that, in view of results in section 4.1, such a choice does not impose any restriction on the dynamic of the model.

of $g \in G_r$:

$$\mathcal{A}^{\hat{K},\,g(\hat{K})}_{\Delta^*_{\varepsilon(p)}} \;=\; \sum_{\hat{I}\in P^1_+} \mathcal{N}^{\hat{I}}_{\hat{K}\,\hat{K}}\,\mathrm{Tr}_{\mathcal{H}_{\hat{I}}}[\tau_h q^{L_0^{(O)}-\frac{r}{24}}],$$

where τ_h is the action induced by the selected group element on $\mathcal{H}_{\hat{I}}$. Hence the full amplitude on a fixed geometry parametrized by a choice of the ribbon graph Γ and of the set of localized curvature assignations $\{\varepsilon(s)\}, s = 1, \dots N_0(T)$, becomes:

$$\begin{aligned}
\mathcal{A}(\Gamma,\{\varepsilon(s)\}) \;=\;& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.2)\\
& N^{2N_0+N_1+N_2} \sum_{\{\hat{I}(\rho^1)\}\in P^1_+(\hat{\mathfrak{g}})} \prod_{\{\rho^0(p,q,r)\}}^{N_2(T)} \begin{Bmatrix} \hat{I}_1(q,p) & \hat{J}_1(p) & \hat{J}_2(q) \\ \hat{J}_3(q) & \hat{I}_2(r,s) & \hat{I}_3(q,r) \end{Bmatrix}_{Q=e^{\frac{2\pi i}{1+h^\vee}}} \\
& \prod_{\{\rho^1(p,r)\}}^{N_1(T)} \left(b^{\hat{J}(p)\hat{J}(r)}_{\hat{I}(r,p)}\right)^2 L(p,r)^{-2H_{\hat{I}(r,p)}} \prod_{s=1}^{N_0(T)} \sum_{\hat{K}\in P^1_+(\hat{\mathfrak{g}})} \mathcal{N}^{\hat{K}}_{\hat{I}(s)\,\hat{I}(s)} \mathrm{Tr}_{\mathcal{H}_{\hat{K}}}[\tau_{h(s)} q^{L_0^{(O)}-\frac{p+1}{24}}] \\
& \prod_{m=1}^{D-p-1} \Bigg[\sum_{\{\hat{j}(\rho^1)\}\in P^1_+(\hat{\mathfrak{su}}(2))} \prod_{\{\rho^0(p,q,r)\}}^{N_2(T)} \begin{Bmatrix} j_1(q,p) & j_1(p) & j_2(q) \\ j_3(q) & j_2(r,s) & j_3(q,r) \end{Bmatrix}_{Q=e^{\frac{2\pi i}{1+h^\vee}}} \\
& \prod_{\{\rho^1(p,r)\}}^{N_1(T)} \left(b^{j(p)j(r)}_{j(r,p)}\right)^2 L(p,r)^{-2H_{j(r,p)}} \prod_{s=1}^{N_0(T)} \sum_{j\in P^1_+(\hat{\mathfrak{su}}(2))} \mathcal{N}^{j}_{j(s)\,j(s)} \mathrm{Tr}_{\mathcal{H}_j}[\tau_{h(s)} q^{L_0^{(O)}-\frac{1}{24}}] \Bigg]_{(m)} ,
\end{aligned}$$

where the factor $N^{2N_0+N_1+N_2}$ takes into account the degeneracy provided by the kinematical $\mathrm{U}(1)^N$ Chan-Paton degrees of freedom.

The above scenario provides hopeful perspectives for its generalization to the non Abelian case, in which the $\mathrm{U}(N)$ symmetry imposed in section 3 is not broken. To this end, a key point will be to look for a new consistent (although equivalent to (3.8)) identification between the background matrix components and the entries of the Cartan matrix of the affine algebra underlying the WZW-model. In particular, we do expect that the extension to the non Abelian case may alter the $N^{2N_0+N1+N2}$ degeneracy factor appearing in (5.2). From a broader perspective, it would be interesting to look for a dictionary between the geometrical parameters underline the full construction in this manuscript and physical quantities of a full-fledged bosonic string theory. We hope to address such issues in a future paper.

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A. Useful formulae

This section contains a collection of useful equations and formulae which can be found in standard group and algebra theory textbooks such as [30, 31, 38]

- **Direct products** If a group G is a direct product of groups $G = G_1 \times G_2$, then, given any two elements $g_1 \in G_1$ and $g_2 \in G_2$, then a representation R of G can be written as
$$R^{\hat{I}_1 \times \hat{I}_2}_{m_1 n_1; m_2 n_2}(g_1 g_2) \;=\; R^{(1)\hat{I}_1}_{m_1 n_1}(g_1)\, R^{(2)\hat{I}_2}_{m_2 n_2}(g_2), \tag{A.1}$$
being $R^{(1)}$ and $R^{(2)}$ a representation respectively of G_1 and G_2.

- **Clebsh-Gordan expansion** Let us consider the expansion of the Kronecker product of two representations:
$$R^{\hat{I}_1}\, R^{\hat{I}_2} \;=\; \sum_{\hat{I} \in P^+_k(\mathfrak{g})} (\hat{I}_1\, \hat{I}_2\, \hat{I}) R^{\hat{I}},$$
where $(\hat{I}_1\, \hat{I}_2\, \hat{I})$ is the number of times that $R^{\hat{I}}$ enters in the Kronecker product of $R^{\hat{I}_1}$ and $R^{\hat{I}_2}$.

  Now let us consider the product of two representation functions with the same argument. It can be expanded in the Clebsh-Gordan series:
$$R^{\hat{I}_1}_{m_1\, n_1}(\Gamma) R^{\hat{I}_2}_{m_2\, n_2}(\Gamma) = \sum_{\hat{I}/(\hat{I}_1\, \hat{I}_2\, \hat{I}) \neq 0} \;\sum_{M,N=1,\ldots,\dim|\hat{I}|} C^{\hat{I}\, M}_{\hat{I}_1\, m_1\, \hat{I}_2\, m_2}\; D^{\hat{I}}_{M\, N}(\Gamma)\, C^{\hat{I}\, N}_{\hat{I}_1\, n_1\, \hat{I}_2\, n_2}\,, \tag{A.2}$$
where the sum is extended to those unitary representations for which the coefficient $(\hat{I}_1\, \hat{I}_2\, \hat{I})$ is non zero.

- **Completeness relations for Clebsh-Gordan coefficients**
$$\sum_{m_1\, m_2} C^{\hat{I}\, m}_{\hat{I}_1\, m_1\, \hat{I}_2\, m_2} C^{\hat{I}'\, m'}_{\hat{I}_1\, m_1\, \hat{I}_2\, m_2} = \delta_{\hat{I}\, \hat{I}'} \delta_{m\, m'}, \tag{A.3a}$$
$$\sum_{\hat{I}\, m} C^{\hat{I}\, m}_{\hat{I}_1\, m_1\, \hat{I}_2\, m_2} C^{\hat{I}\, m}_{\hat{I}_1\, m_1'\, \hat{I}_2\, m_2'} = \delta_{m_1\, m_1'} \delta_{m_2\, m_2'}. \tag{A.3b}$$

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